Permutations and Combinations

Permutations

A permutation is a re-ordering of a set of symbols or objects. For example, the set {a, b, c} can re-ordered like this:

a b c	b a c	c a b
a c b	b c a	c b a

A set { a, b, c, d } can be re-ordered in the following ways:

a b c d	b a c d	c a b d	d a b c
a b d c	b a d c	c a d b	d a c b
a c b d	b c a d	c b a d	d b a c
a c d b	b c d a	c b d a	d b c a
a d b c	b d a c	c d a b	d c a b
a d c b	b d c a	c d b a	d c b a

As you can see, the number of permutations for a set of 4 items is significantly larger than for a set of 3 items. (6 compared to 24 in fact.) How can you determine how many permutations there would be for any given set?

Consider the set of three items. The first item in the list can be one of three items. With the one of the items used, the second item can be any of the remaining 2. And the last can only be the one remaining. The number of permutations would be the product of each option, or:

3 * 2 * 1 (= 6)

In the case of the set of 4 items, we can apply the same idea to get the following product:

4 * 3 * 2 * 1 (= 24)

This matches the count we got when we listed all of the options explicitly.
A way to write this uses the following notation:

$^{n}\mathrm{P}$

where *n* is the number of items to be re-ordered.

A second format is easier to type, and uses function syntax:

P(*n*)

Factorial

The number of re-ordered set can be determined using *factorials*. A factorial is the product of a value from a starting value to an end value (usually 1). For example, the factorial of 6 would be:

6 * 5 * 4 * 3 * 2 * 1 (= 720)

The symbol commonly used is the exclamation mark (!). Thus, 6! = 720.

But what if you only wanted sets of 3 from a set of 4? For example, if we wanted sets of 3 letters from our set of 4 letters, how would we figure that out? Let's take a look at an explicit list:

a b c	b a c	c a b	d a b
a b d	b a d	c a d	d a c
a c b	b c a	c b a	d b a
a c d	b c d	c b d	d b c
a d b	b d a	c d a	d c a
a d c	b d c	c d b	d c b

At first glance, the number of options didn't change, so maybe they *are* the same. But that doesn't make sense. Consider how we figured out how many sets of 4 we could get from a set of four:

4 * 3 * 2 *1

The last '1' mean that there was still one left. But if we lop off the '1', that doesn't change the product – in *this* case. Let's take a look at lists of 2 items from a set of 4:

a b	b a	c a	d a
a c	b c	c b	d b
a d	b d	c d	d c

This is much smaller. Using our previous technique, we can determine that the product should be

4 * 3 (= 12)

There are two ways to think of this. One is that you simply use only the first *n* values in the product (where *n* is the number of items we want in our subsets). Thus, from 4! We want only sets of 2, we multiply 4 * 3, and ignore the rest. The notation for this is:

$^{n}P_{r}$

where n is the number of items in the original list, and r is the number of items to be selected from the list.

The other notation form looks like this:

$$P(n, r)$$

A mathematically rigorous description using n and r is:

$$P(n, r) = \frac{n!}{(n - r)!}$$

Using our example of sets of 2 items from a set of 4,

$$P(4, 2) = \frac{4!}{(4 - 2)!} = \frac{24}{2} = 12$$

An important concept to remember is that permutation are order sensitive. That means that order matters. Set *abc* is not the same as *bac*.

Combinations

What if we want sets of items, but order *doesn't* matter? Examples include members of a work group or committee, or items in a grocery bag. There may be 6 items of your total purchase of 9 items, but the *order* of those 6 items doesn't matter.

Let's take a look at our first set of { a, b, c }, and select just two. Let's see what we have:

a b	b a	c a
a c	b c	c b

Delete the ones that have the same letters. Let's just look at set *ab* first.

a b	~~b a~~	c a
a c	b c	c b

Using the same principle on the others, the results are reduced:

a b	a c	b c

So from the original 6 we now get 3.

How about some other quantities? Let's take a look at set {a, b, c, d }. How can we get subsets of three items out of the four? Let's see our solution using permutations:

a b c	b a c	c a b	d a b
a b d	b a d	c a d	d a c
a c b	b c a	c b a	d b a
a c d	b c d	c b d	d b c
a d b	b d a	c d a	d c a
a d c	b d c	c d b	d c b

Now, let's delete the ones that have the same letters. Let's just look at set *abc* first.

a b c	~~b a c~~	~~c a b~~	d a b
a b d	b a d	c a d	d a c
~~a c b~~	~~b c a~~	~~c b a~~	d b a
a c d	b c d	c b d	d b c
a d b	b d a	c d a	d c a
a d c	b d c	c d b	d c b

So it looks like we will have reduced the number of subsets to one sixth (by applying the same principle to the other subsets). We trim the original 24 to just 4 and end up with the following:

a b c	a b d	a c d	b c d

A pattern is starting to emerge.

If we define a function, C, with two parameters: C(*n, r*), where *n* is the number of elements in the original set, and *r* is the number of items to be selected. Let's look at our examples:

$n = 3 \qquad r = 2 \qquad C(3, 2) = \dfrac{P(n,r)}{2} = \dfrac{P(3,2)}{2} = \dfrac{6}{2} = 3$

and

$n = 4 \qquad r = 3 \qquad C(4, 3) = \dfrac{P(n,r)}{6} = \dfrac{P(4,3)}{6} = \dfrac{24}{6} = 4$

The source for the denominator is the factorial of the second term, *r*, which can expressed as *r*!. Thus, finding the number of combinations can be described like this:

$$C(n, r) = \frac{P(n,r)}{r!} = \frac{\left(\dfrac{n!}{(n-r)!}\right)}{r!} = \frac{n!}{(n-r)!r!}$$

One of these terms should make sense to you. Another notation for combinations is shown here:

$^{n}C_{r}$

Probability

One use for probability is in calculating warranty cost. For example, if a solution for a periodic manufacturing defect costs $1000, and it cost $10 to correct the defect when it is found by the customer, it would make sense to correct the defect in manufacturing only if you expect to ship more than 100 defective items.

But the question then becomes: will the company sell 10 defective items? That depends. The more important question becomes: how often are defective items shipped?, or what is the chance or *probability* of shipping a defective item?

The probability of any event can be determined by creating a ratio between the number of 'successful' events and total number of possible, or 'sample', events.

$$P(E) = \frac{|E|}{|S|}$$

[Note: The use of the brackets indicates the *size of a set* rather than an absolute value.] [Also note that the *P*( ) function is overloaded when compared with permutations and other possible functions. This is a convention in the field, and not my doing or preference.]

For example, a six sided die has six (6) possible sides. If we want to know the probability of rolling a 4, we note that there is only one successful event, or E. The possible 'sample' size, or S, is six (6). Therefore, the probability of rolling a 4 is 1/6.

What is the probability of rolling an even number? The number of 'successful' events is 3 (the number of even numbers from 1 to 6), divided by the sample size of 6. Thus, the probability is 3/6.