# The Exponential Shift

Occasionally, a differential equation with constant coefficients appears that would be simpler to solve if the forcing function did not have an exponential. In these cases, the exponential shift can transform the equation into one that does not have that exponential.

First, let's prove the exponential shift theorem:

$$e^{ax} f(D)y = f(D-a)[e^{ax} y]$$

To start, let's evaluate $(D-a)[e^{ax} y]$.

$$\begin{aligned}
(D-a)[e^{ax} y] &= D[e^{ax} y] - a[e^{ax} y] \\
&= e^{ax}D\, y + y\, De^{ax} - ae^{ax}\, y \\
&= e^{ax}D\, y + y\, ae^{ax} - ae^{ax}\, y \\
&= e^{ax}D\, y
\end{aligned}$$

Next, let's evaluate $(D-a)^2[e^{ax} y]$.

$$\begin{aligned}
(D-a)^2[e^{ax} y] &= (D-a)\,(D-a)\,[e^{ax} y] \\
&= (D-a)\,[e^{ax} Dy] \\
&= D\,[e^{ax} Dy] - a[e^{ax} Dy] \\
&= e^{ax} D^2 y + ae^{ax} Dy - ae^{ax} Dy \\
&= e^{ax} D^2 y
\end{aligned}$$

From here, induction will show that $(D-a)^n[e^{ax} y] = e^{ax} D^n y$ is true for all $n$ greater than one.

If $(D-a)^k[e^{ax} y] = e^{ax} D^k y$, then:

$$\begin{aligned}
(D-a)^{k+1}[e^{ax} y] &= (D-a)\,(D-a)^k\,[e^{ax} y] \\
&= (D-a)\,[e^{ax} D^k y] \\
&= D\,[e^{ax} D^k y] - a[e^{ax} D^k y] \\
&= e^{ax} D^{k+1} y + ae^{ax} D^k y - ae^{ax} D^k y \\
&= e^{ax} D^{k+1} y
\end{aligned}$$

If $f$ is a polynomial, then $f(D)$ is just a linear combination of $D^n$ terms. The fact that $(D - a)^n[e^{ax} y] = e^{ax} D^n y$ would hold for each term in the equation to be proved:

$$e^{ax} f(D)y = f(D - a)[e^{ax} y]$$

Let's use this in some examples. We will only use the exponential shift to help find the particular solution.

**Example 1**

$$[D^2 - 2D + 5]\, y = 16x^3e^{3x} \tag{1}$$

We will treat the homogeneous solution, which is $y_h = e^x (A\cos 2x + B\sin 2x)$, separately.

$$[D^2 - 2D + 5]\, y_p = 16x^3e^{3x} \tag{2}$$

$$e^{-3x} [D^2 - 2D + 5]\, y_p = 16x^3 \tag{3}$$

$$[(D + 3)^2 - 2(D + 3) + 5]\, (e^{-3x} y_p) = 16x^3 \tag{4}$$

$$[D^2 + 4D + 8]\, (e^{-3x} y_p) = 16x^3 \tag{5}$$

From here, we can say that $(e^{-3x} y_p) = Cx^3 + Ex^2 + Fx + G$. We can find $y_p$ in a familiar way:

$$(e^{-3x} y_p) = Cx^3 + Ex^2 + Fx + G \tag{6}$$

$$D(e^{-3x} y_p) = 3Cx^2 + 2Ex + F \tag{7}$$

$$D^2(e^{-3x} y_p) = 6Cx + 2E \tag{8}$$

Substitution gives the following:

$$[D^2 + 4D + 8]\, (e^{-3x} y_p) = (6Cx + 2E) + 4(3Cx^2 + 2Ex + F) +8(Cx^3 + Ex^2 + Fx + G) \tag{9}$$

$$= 8Cx^3 + (12C + 8E)x^2 + (6C + 8E + 8F)x + 2E + 4F + 8G \tag{10}$$

From which we find that: $C = 2$, $E = -3$, $F = 3/2$, $G = 0$, and:

$$e^{-3x} y_p = 2x^3 - 3x^2 + \tfrac{3}{2}x \tag{11}$$

$$y_p = \left(2x^3 - 3x^2 + \tfrac{3}{2}x\right)e^{3x} \tag{12}$$

The general solution is then:

$$y = y_h + y_p = e^x\left(A\cos 2x + B\sin 2x\right) + \left(2x^3 - 3x^2 + \tfrac{3}{2}x\right)e^{3x} \tag{13}$$

**Example 2**

$$[D^2 - 2D + 1]\, y = xe^x + 7x - 2 \tag{14}$$

In this case, the homogeneous solution is: $y_h = e^x(A + Bx)$. We would expect that the particular solution would have this form: $y_p = e^x(Cx^2 + Ex^3) + Fx + G$. Here, we will use superposition and split the particular solution into two pieces, the first of which we will find using the exponential shift.

$$y_p = y_{p1} + y_{p2} \tag{15}$$

$$[D^2 - 2D + 1]\, y_{p1} = xe^x \tag{16}$$

$$[D^2 - 2D + 1]\, y_{p2} = 7x - 2 \tag{17}$$

Solving (16):

$$[D^2 - 2D + 1]\, y_{p1} = xe^x \tag{18}$$

$$e^{-x}\, [D^2 - 2D + 1]\, y_{p1} = x \tag{19}$$

$$[(D + 1)^2 - 2(D + 1) + 1]\, (e^{-x}\, y_{p1}) = x \tag{20}$$

$$D^2\, (e^{-x}\, y_{p1}) = x \tag{21}$$

We can think of this as being $D^2v = x$, where $v = e^{-x}y$. From this, we could say that $v_h = C_1x + C_2$, and that $v_p = C_3x^3 + C_4x^2$. Note that the two statements, $v_h = C_1x + C_2$ and $y_h = e^x(A + Bx)$, are equivalent. It is a straightforward thing to show that $v_p = \frac{1}{6}x^3$. So:

$$e^{-x}\, y_{p1} = \tfrac{1}{6}x^3 \tag{22}$$

$$y_{p1} = \tfrac{1}{6}x^3e^x \tag{23}$$

For (17), it is clear that $y_{p2} = C_5x + C_6$, and:

$$[D^2 - 2D + 1]\, y_{p2} = 7x - 2 \tag{24}$$

$$0 - 2(C_5) + C_5x + C_6 = 7x - 2 \tag{25}$$

$$C_5x + (C_6 - 2C_5) = 7x - 2 \tag{26}$$

From this, we find that $C_5 = 7$ and $C_6 = 12$. Putting this all together, we have:

$$y = y_h + y_p = e^x\left(A + Bx\right) + \tfrac{1}{6}x^3e^x + 7x + 12 \tag{27}$$

This was time consuming, but it might have been more so if $y_{p1}$ were found without using the exponential shift.

Reference: Elementary Differential Equations, 5th edition, Earl D. Rainville and Phillip E. Bedient