Formulas for Installment Payments

The MAT 102 textbook tells you to use a table to look up the finance charge per \$100 for installment loans. The following formulas can be used to determine the payment amount for a fixed payment installment loan.

The following variables represent the following quantities:

n = the number of payments per year

r = the rate per year, expressed as a decimal

T = the term, or duration of the loan, in years

$N = nT$; the total number of payments

p = payment amount

P = principal

$x = \frac{r}{n}$; this is to simplify the equations

$B(k)$ = balance due after k payments

$$A = P(1+x)^N = P\left(1+\frac{r}{n}\right)^N = P\left(1+\frac{r}{n}\right)^{nT}$$

For a conventional loan:

$$p = P\,\frac{x(1+x)^N}{(1+x)^N - 1} = P\,\frac{\frac{r}{n}\left(1+\frac{r}{n}\right)^N}{\left(1+\frac{r}{n}\right)^N - 1} = P\,\frac{\frac{r}{n}\left(1+\frac{r}{n}\right)^{nT}}{\left(1+\frac{r}{n}\right)^{nT} - 1}$$

$$B(k) = P\,\frac{(1+x)^{N+1} - (1+x)^{N-k+1}}{(1+x)^N - 1} = P\,\frac{\left(1+\frac{r}{n}\right)^{N+1} - \left(1+\frac{r}{n}\right)^{N-k+1}}{\left(1+\frac{r}{n}\right)^N - 1} = P\,\frac{\left(1+\frac{r}{n}\right)^{nT+1} - \left(1+\frac{r}{n}\right)^{nT-k+1}}{\left(1+\frac{r}{n}\right)^{nT} - 1}$$

or:

$$B(k) = P\,\frac{(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{(1+x)^N - 1} = P\,\frac{\left(1+\frac{r}{n}\right)^{N-k+1}\left(\left(1+\frac{r}{n}\right)^k - 1\right)}{\left(1+\frac{r}{n}\right)^N - 1} = P\,\frac{\left(1+\frac{r}{n}\right)^{nT-k+1}\left(\left(1+\frac{r}{n}\right)^k - 1\right)}{\left(1+\frac{r}{n}\right)^{nT} - 1}$$

For example, look at a loan of $100,000 at a rate of 6% per year for 30 years with monthly payments:

$$p = 100{,}000 \frac{\frac{0.06}{12}\left(1+\frac{0.06}{12}\right)^{12\times 30}}{\left(1+\frac{0.06}{12}\right)^{12\times 30} - 1} = 100{,}000 \frac{0.005(1+0.005)^{360}}{(1+0.005)^{360}-1} = 100{,}000 \frac{0.005 \times 6.022575212263}{6.022575212263 - 1}$$

$$p = 100{,}000 \frac{0.03011287606131}{5.022575212263} = 599.5505251528 \approx \$599.55$$

If you were to pay the loan according to the schedule, you would pay a total of:

Installment cost: 599.55 × 360 = $215,838.00

After 120 payments, your balance would be:

$$B = 100{,}000 \frac{\left(1+\frac{0.06}{12}\right)^{361} - \left(1+\frac{0.06}{12}\right)^{241}}{\left(1+\frac{0.06}{12}\right)^{360} - 1} = 100{,}000 \frac{6.052688088324 - 3.326755498186}{6.022575212263 - 1} \approx \$54{,}273.60$$

Keep in mind that after 120 payments, you still owe 240 payments of $599.55, which adds up to $143,892.00. The difference, $89,618.41, is the interest that you will accrue on the unpaid balance over those 20 years.

Loans with shorter terms have larger payments, but the amount paid ends up being less.

Taking the above example, if a 15 year mortgage were taken at the same annual rate with monthly payments:

$$p = 100{,}000 \frac{\frac{0.06}{12}\left(1+\frac{0.06}{12}\right)^{12\times 15}}{\left(1+\frac{0.06}{12}\right)^{12\times 15} - 1} = 100{,}000 \frac{0.005(1+0.005)^{180}}{(1+0.005)^{180}-1} = 100{,}000 \frac{0.005 \times 2.454093562247}{2.454093562247 - 1}$$

$$p = 100{,}000 \frac{0.01227046781124}{1.454093562247} = 843.8568280486 \approx \$843.86$$

The installment cost would be: 843.86 × 180 = $151,894.80. The installment cost is less for the 15 year mortgage than for a 30 year mortgage with the same interest rate, but the payments are larger.

The derivations for the given equations is based on a description of how the balance changes over time, until it finally reaches zero. Initially, the balance is the principal. Then interest is added and payments are subtracted in successive steps. The balance after n payments is denoted as $B(n)$:

$$B(0) = P$$

$$B(1) = P(1+x) - p$$

$$B(2) = [P(1+x) - p](1+x) - p$$

Finally, after the final payment, the balance is zero.:

$$B(N) = \{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) - p = 0$$

Successively adding p to both sides and dividing both sides by $(1 + x)$ yields an equation that has the principal P on one side, and an expression containing the payment p on the other.:

$$\{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) - p = 0$$

$$\{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) = p$$

$$\{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) - p = p\frac{1}{1+x}$$

$$\{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) = p\frac{1}{1+x} + p$$

$$\{\cdots[P(1+x) - p](1+x) - p\cdots\}(1+x) - p = \left(p\frac{1}{1+x} + p\right)\frac{1}{1+x} = p\left(\frac{1}{1+x} + \left(\frac{1}{1+x}\right)^2\right)$$

Continuing this process to the end yields this equation:

$$P = p\left(\frac{1}{1+x} + \left(\frac{1}{1+x}\right)^2 + \left(\frac{1}{1+x}\right)^3 + \cdots + \left(\frac{1}{1+x}\right)^N\right)$$

Let's make a substitution to keep the equations simple. Let:

$$z = \frac{1}{1+x}$$

So that:

$$P = p\left(z + z^2 + z^3 + \cdots + z^N\right) = p\sum_{n=1}^{N} z^n$$

The summation symbol, Σ, is used to simplify the equation further.

The fact that the absolute value of z is less than one allows us to use the following relationship to express the relationship between the payment and principal in closed form.:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad \text{if } |z| < 1$$

Since z is the reciprocal of a number that is slightly larger than one, we know that we can use this relation. What follows are commonly used manipulation techniques.:

$$P = p\sum_{n=1}^{N} z^n$$

$$P = pz\sum_{n=0}^{N-1} z^n$$

$$P = pz\left(\left(\sum_{n=0}^{\infty} z^n\right) - \left(\sum_{n=N}^{\infty} z^n\right)\right)$$

$$P = pz\left(\left(\sum_{n=0}^{\infty} z^n\right) - z^N\left(\sum_{n=0}^{\infty} z^n\right)\right)$$

$$P = pz\left(1 - z^N\right)\left(\sum_{n=0}^{\infty} z^n\right)$$

$$P = \frac{pz\left(1 - z^N\right)}{1-z}$$

$$P = \frac{p\left(z - z^{N+1}\right)}{1-z}$$

Now that we have a closed-form equation relating the principal to the payment, we can convert it to a form that has the variables we acutually use. First, we will express the payment in terms of the principal, then replace z with $1/(1 + x)$, then simplify:

$$p = P\frac{1-z}{z - z^{N+1}}$$

$$p = P\left(\frac{1 - \frac{1}{1+x}}{\frac{1}{1+x} - \left(\frac{1}{1+x}\right)^{N+1}}\right)$$

$$p = P\left(\frac{1 - \frac{1}{1+x}}{\frac{1}{1+x} - \left(\frac{1}{1+x}\right)^{N+1}}\right)\left(\frac{(1+x)^{N+1}}{(1+x)^{N+1}}\right)$$

$$p = P\frac{(1+x)^{N+1} - (1+x)^N}{(1+x)^N - 1} = P\frac{(1+x)^N(1+x-1)}{(1+x)^N - 1} = P\frac{x(1+x)^N}{(1+x)^N - 1}$$

Finding the expression for the balance after k payments follows a similar path. Using our expressions for $B(n)$, we can find the balance when $n = k$.:

$$B(k) = p\left(1 + \frac{1}{1+x} + \left(\frac{1}{1+x}\right)^2 + \cdots \left(\frac{1}{1+x}\right)^{k-1}\right)$$

$$B(k) = p\sum_{n=0}^{k-1}\left(\frac{1}{1+x}\right)^n$$

$$B(k) = p\sum_{n=0}^{k-1} z^n$$

Using the manipulation methods used above:

$$B(k) = p\sum_{n=0}^{k-1} z^n$$

$$B(k) = p\,\frac{1-z^k}{1-z}$$

Now we substitute our expression for p to get an equation in terms of P.:

$$B(k) = p\left(\frac{1-z^k}{1-z}\right)$$

$$B(k) = P\left(\frac{x(1+x)^N}{(1+x)^N - 1}\right)\left(\frac{1-z^k}{1-z}\right)$$

$$B(k) = P\left(\frac{x(1+x)^N}{(1+x)^N - 1}\right)\left(\frac{1-\left(\frac{1}{1+x}\right)^k}{1-\frac{1}{1+x}}\right)$$

$$B(k) = P\left(\frac{x(1+x)^N}{(1+x)^N - 1}\right)\left(\frac{1-\left(\frac{1}{1+x}\right)^k}{1-\frac{1}{1+x}}\right)\frac{(1+x)^k}{(1+x)^k}$$

$$B(k) = P\left(\frac{x(1+x)^N}{(1+x)^N - 1}\right)\left(\frac{(1+x)^k - 1}{(1+x)^k - (1+x)^{k-1}}\right)$$

From here, we multiply the denominator factors out to get a simpler expression.:

$$B(k) = P\left(\frac{x(1+x)^N}{(1+x)^N - 1}\right)\left(\frac{(1+x)^k - 1}{(1+x)^k - (1+x)^{k-1}}\right)$$

$$B(k) = P\frac{x(1+x)^N\left((1+x)^k - 1\right)}{(1+x)^{N+k} - (1+x)^{N+k-1} - (1+x)^k + (1+x)^{k-1}}$$

$$B(k) = P\frac{x(1+x)^N\left((1+x)^k - 1\right)}{(1+x)^{k-1}\left((1+x)^{N+1} - (1+x)^N - (1+x) + 1\right)}$$

$$B(k) = P\frac{x(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{(1+x)^{N+1} - (1+x)^N - (1+x) + 1}$$

$$B(k) = P\frac{x(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{(1+x)^{N+1} - (1+x)^N - x}$$

$$B(k) = P\frac{x(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{(1+x)^N(1+x-1) - x}$$

$$B(k) = P\frac{x(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{x(1+x)^N - x}$$

$$B(k) = P\frac{x(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{x\left((1+x)^N - 1\right)}$$

$$B(k) = P\frac{(1+x)^{N-k+1}\left((1+x)^k - 1\right)}{(1+x)^N - 1}$$

$$B(k) = P\frac{(1+x)^{N+1} - (1+x)^{N-k+1}}{(1+x)^N - 1}$$

The last two equations are the given equations for the balance.

Remember that these expressions do not account for the effects of rounding every time that interest is added. They are, however, as good as you can get with closed-form expressions.