Section 12.3
Systems of Linear Equations: Determinants
OBJECTIVE 1

1 ✔ Evaluate 2 by 2 Determinants
If $a$, $b$, $c$, and $d$ are four real numbers, the symbol

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a **2 by 2 determinant**. Its value is the number $ad - bc$; that is,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]
Evaluate:

\[
\begin{vmatrix}
-2 & 3 \\
4 & -1
\end{vmatrix}
\]
OBJECTIVE 2

2. Use Cramer’s Rule to Solve a System of Two Equations Containing Two Variables
Cramer’s Rule for Two Equations Containing Two Variables

The solution to the system of equations

\[
\begin{align*}
ax + by &= s \\
 cx + dy &= t
\end{align*}
\]

is given by

\[
x = \frac{s \begin{vmatrix} t & d \\ b & c \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}
\]

provided that

\[
D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0
\]
Cramer’s Rule

\[
\begin{align*}
\begin{cases}
ax + by &= s \\
 cx + dy &= t
\end{cases}
\quad D &= \begin{vmatrix} a & b \\
 c & d \end{vmatrix} \\
D_x &= \begin{vmatrix} s & b \\
 t & d \end{vmatrix} \\
D_y &= \begin{vmatrix} a & s \\
 c & t \end{vmatrix}
\end{align*}
\]

if \( D \neq 0 \),

\[
\begin{align*}
x &= \frac{D_x}{D}, & y &= \frac{D_y}{D}
\end{align*}
\]
EXAMPLE

Solving a System of Linear Equations Using Determinants

Use Cramer's Rule, if applicable, to solve the system

\[
\begin{align*}
3x - 6y &= 24 \\
5x + 4y &= 12
\end{align*}
\]

\[x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}\]
OBJECTIVE 3

3 Evaluate 3 by 3 Determinants
A 3 by 3 determinant is symbolized by

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

in which \(a_{11}, a_{12}, \ldots\), are real numbers.

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33}
\end{vmatrix} - a_{12} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix} + a_{13} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix}
\]

2 by 2 determinant left after removing row and column containing \(a_{11}\)

2 by 2 determinant left after removing row and column containing \(a_{12}\)

2 by 2 determinant left after removing row and column containing \(a_{13}\)
EXAMPLE

Finding Minors of a 3 by 3 Determinant

For the determinant \( A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix} \),

Find: (a) \( M_{21} \) \quad (b) \( M_{32} \)
For an \( n \) by \( n \) determinant \( A \), the **cofactor** of entry \( a_{ij} \), denoted by \( A_{ij} \), is given by

\[
A_{ij} = (-1)^{i+j} M_{ij}
\]

where \( M_{ij} \) is the minor of entry \( a_{ij} \).
EXAMPLE

Evaluating a $3 \times 3$ Determinant

Find the value of the $3$ by $3$ determinant:

$$
\begin{vmatrix}
1 & 2 & 1 \\
3 & 5 & 1 \\
2 & 6 & 7
\end{vmatrix}
$$
OBJECTIVE 4

4 Use Cramer’s Rule to Solve a System of Three Equations Containing Three Variables
\[
\begin{align*}
\begin{cases}
 a_{11}x + a_{12}y + a_{13}z &= c_1 \\
 a_{21}x + a_{22}y + a_{23}z &= c_2 \\
 a_{31}x + a_{32}y + a_{33}z &= c_3 
\end{cases}
\end{align*}
\]

\[D = \begin{vmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{vmatrix} \neq 0\]

\[x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}\]

Cramer’s Rule for Three Equations Containing Three Variables

\[D_x = \begin{vmatrix}
 c_1 & a_{12} & a_{13} \\
 c_2 & a_{22} & a_{23} \\
 c_3 & a_{32} & a_{33}
\end{vmatrix}, \quad D_y = \begin{vmatrix}
 a_{11} & c_1 & a_{13} \\
 a_{21} & c_2 & a_{23} \\
 a_{31} & c_3 & a_{33}
\end{vmatrix}, \quad D_z = \begin{vmatrix}
 a_{11} & a_{12} & c_1 \\
 a_{21} & a_{22} & c_2 \\
 a_{31} & a_{32} & c_3
\end{vmatrix}\]
Use Cramer's Rule, if applicable, to solve the following system:

\[
\begin{align*}
    x + 2y + z &= 1 \\
    3x + 5y + z &= 3 \\
    2x + 6y + 7z &= 1
\end{align*}
\]

\[
    x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}
\]
Cramer’s Rule with Inconsistent or Dependent Systems

- If $D = 0$ and at least one of the determinants $D_x$, $D_y$, or $D_z$ is different from 0, then the system is inconsistent and the solution set is $\emptyset$ or $\{\}$.
- If $D = 0$ and all the determinants $D_x$, $D_y$, and $D_z$ equal 0, then the system is consistent and dependent so that there are infinitely many solutions. The system must be solved using row reduction techniques.
OBJECTIVE 5

5 Know Properties of Determinants
Properties of Determinants

The value of a determinant changes sign if any two rows (or any two columns) are interchanged.

\[ \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \]
\[ \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \]

Demonstrating Theorem (11)
Properties of Determinants

If all the entries in any row (or any column) equal 0, the value of the determinant is 0. \[(12)\]

If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0. \[(13)\]

**Demonstrating Theorem (13)**

\[
\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\
= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0
\]
If any row (or any column) of a determinant is multiplied by a nonzero number \(k\), the value of the determinant is also changed by a factor of \(k\). \((14)\)

**Demonstrating Theorem (14)**

\[
\begin{vmatrix}
1 & 2 \\
4 & 6 \\
\end{vmatrix} = 6 - 8 = -2
\]

\[
\begin{vmatrix}
k & 2k \\
4 & 6 \\
\end{vmatrix} = 6k - 8k = -2k = k(-2) = k
\begin{vmatrix}
1 & 2 \\
4 & 6 \\
\end{vmatrix}
\]
Properties of Determinants

If the entries of any row (or any column) of a determinant are multiplied by a nonzero number $k$ and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged. (15)

Demonstrating Theorem (15)

$$
\begin{vmatrix}
3 & 4 \\
5 & 2
\end{vmatrix} = -14
$$

$$
\begin{vmatrix}
3 & 4 \\
5 & 2
\end{vmatrix} \rightarrow
\begin{vmatrix}
-7 & 0 \\
5 & 2
\end{vmatrix} = -14
$$

Multiply row 2 by $-2$ and add to row 1.