# Section 12.3

# Systems of Linear Equations: Determinants

# OBJECTIVE 1

1 Evaluate 2 by 2 Determinants

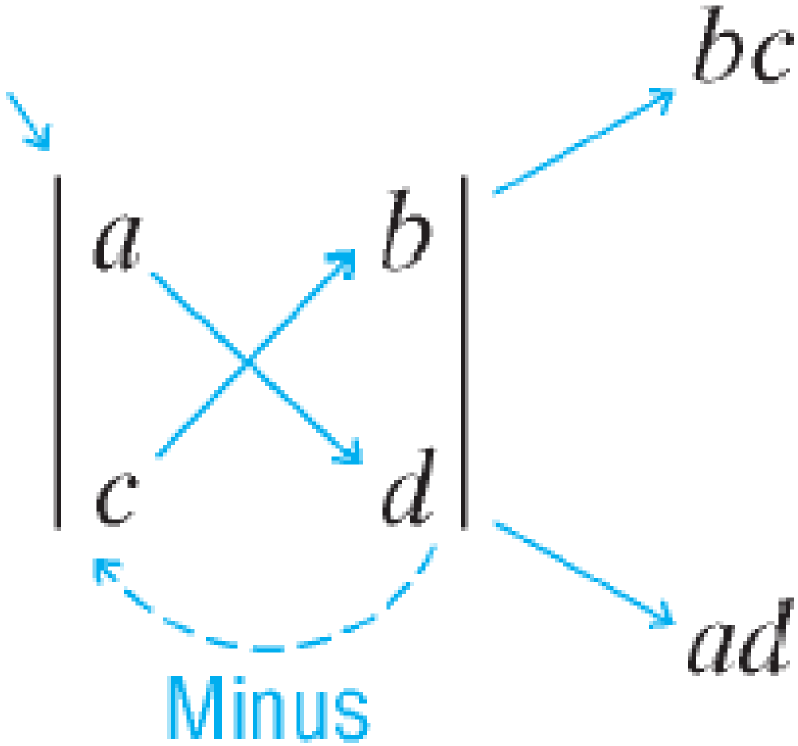
# DEFINITION

If $a$, $b$, $c$, and $d$ are four real numbers, the symbol

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a **2 by 2 determinant**. Its value is the number $ad - bc$; that is,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$




$$= ad - bc$$

EXAMPLE

## Evaluating a 2 × 2 Determinant

Evaluate: $\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix}$

# OBJECTIVE 2

2 Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables

**Cramer's Rule for Two Equations Containing Two Variables**

The solution to the system of equations

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

is given by

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided that

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

# Cramer's Rule

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases} \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix} \qquad D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

if $D \neq 0$,

$$x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D}$$

# EXAMPLE

## Solving a System of Linear Equations Using Determinants

Use Cramer's Rule, if applicable, to solve the system

$$\begin{cases} 3x - 6y = 24 \\ 5x + 4y = 12 \end{cases}$$

$$x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D}$$

# OBJECTIVE 3

3 Evaluate 3 by 3 Determinants

A **3 by 3 determinant** is symbolized by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

in which $a_{11}, a_{12}, \ldots,$ are real numbers.

Minus

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2 by 2 determinant left after removing row and column containing $a_{11}$

2 by 2 determinant left after removing row and column containing $a_{12}$

2 by 2 determinant left after removing row and column containing $a_{13}$

EXAMPLE

## Finding Minors of a 3 by 3 Determinant

For the determinant $A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix}$,

Find: (a) $M_{21}$ (b) $M_{32}$

For an $n$ by $n$ determinant $A$, the **cofactor** of entry $a_{ij}$, denoted by $A_{ij}$, is given by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where $M_{ij}$ is the minor of entry $a_{ij}$.

EXAMPLE

## Evaluating a 3 × 3 Determinant

Find the value of the 3 by 3 determinant: $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{vmatrix}$

# OBJECTIVE 4

4 Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

**Cramer's Rule for Three Equations Containing Three Variables**

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

EXAMPLE **Using Cramer's Rule**

Use Cramer's Rule, if applicable, to solve the following system:

$$\begin{cases} x + 2y + z = 1 \\ 3x + 5y + z = 3 \\ 2x + 6y + 7z = 1 \end{cases}$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

## Cramer's Rule with Inconsistent or Dependent Systems

- If $D = 0$ and at least one of the determinants $D_x$, $D_y$, or $D_z$ is different from 0, then the system is inconsistent and the solution set is $\varnothing$ or $\{\ \}$.
- If $D = 0$ and all the determinants $D_x$, $D_y$, and $D_z$ equal 0, then the system is consistent and dependent so that there are infinitely many solutions. The system must be solved using row reduction techniques.

# OBJECTIVE 5

5 Know Properties of Determinants

# Properties of Determinants

The value of a determinant changes sign if any two rows (or any two columns) are interchanged. **(11)**

## Demonstrating Theorem (11)

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

# Properties of Determinants

If all the entries in any row (or any column) equal 0, the value of the determinant is 0. **(12)**

If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0. **(13)**

## Demonstrating Theorem (13)

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

# Properties of Determinants

If any row (or any column) of a determinant is multiplied by a nonzero number $k$, the value of the determinant is also changed by a factor of $k$. **(14)**

## Demonstrating Theorem (14)

$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 6 - 8 = -2$$

$$\begin{vmatrix} k & 2k \\ 4 & 6 \end{vmatrix} = 6k - 8k = -2k = k(-2) = k\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$$

# Properties of Determinants

If the entries of any row (or any column) of a determinant are multiplied by a nonzero number $k$ and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged. **(15)**

## Demonstrating Theorem (15)

$$\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = -14 \qquad \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} -7 & 0 \\ 5 & 2 \end{vmatrix} = -14$$

Multiply row 2 by $-2$ and add to row 1.