Section 12.4
Matrix Algebra
A **matrix** is defined as a rectangular array of numbers:

<table>
<thead>
<tr>
<th></th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column $j$</th>
<th>Column $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td>$a_{11}$</td>
<td>$a_{12}$</td>
<td>$\ldots$</td>
<td>$a_{1j}$</td>
</tr>
<tr>
<td>Row 2</td>
<td>$a_{21}$</td>
<td>$a_{22}$</td>
<td>$\ldots$</td>
<td>$a_{2j}$</td>
</tr>
<tr>
<td>\vdots</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>Row $i$</td>
<td>$a_{i1}$</td>
<td>$a_{i2}$</td>
<td>$\ldots$</td>
<td>$a_{ij}$</td>
</tr>
<tr>
<td>\vdots</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>Row $m$</td>
<td>$a_{m1}$</td>
<td>$a_{m2}$</td>
<td>$\ldots$</td>
<td>$a_{mj}$</td>
</tr>
</tbody>
</table>
In a survey of 900 people, the following information was obtained:

- 200 males thought federal defense spending was too high.
- 150 males thought federal defense spending was too low.
- 45 males had no opinion.
- 315 females thought federal defense spending was too high.
- 125 females thought federal defense spending was too low.
- 65 females had no opinion.

We can arrange these data in a rectangular array as follows:

<table>
<thead>
<tr>
<th></th>
<th>Too High</th>
<th>Too Low</th>
<th>No Opinion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>200</td>
<td>150</td>
<td>45</td>
</tr>
<tr>
<td>Female</td>
<td>315</td>
<td>125</td>
<td>65</td>
</tr>
</tbody>
</table>

or as the matrix

\[
\begin{bmatrix}
200 & 150 & 45 \\
315 & 125 & 65
\end{bmatrix}
\]

This matrix has two rows (representing males and females) and three columns (representing “too high,” “too low,” and “no opinion”).
EXAMPLE

Examples of Matrices

(a) \[
\begin{bmatrix}
5 & 0 \\
-6 & 1
\end{bmatrix}
\] A 2 by 2 square matrix

(b) \[
\begin{bmatrix}
1 & 0 & 3
\end{bmatrix}
\] A 1 by 3 matrix

(c) \[
\begin{bmatrix}
6 & -2 & 4 \\
4 & 3 & 5 \\
8 & 0 & 1
\end{bmatrix}
\] A 3 by 3 square matrix
OBJECTIVE 1

1. Find the Sum and Difference of Two Matrices
Two $m$ by $n$ matrices $A$ and $B$ are said to be equal, written as

$$A = B$$

provided that each entry $a_{ij}$ in $A$ is equal to the corresponding entry $b_{ij}$ in $B$. 
Adding and Subtracting Matrices

\[ A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -4 \end{bmatrix} \]

Find:

(a) \[ A + B \]
(b) \[ A - B \]
Commutative Property of Matrix Addition

\[ A + B = B + A \]

Associative Property of Matrix Addition

\[(A + B) + C = A + (B + C)\]
EXAMPLE

Demonstrating the Commutative Property

\[
\begin{bmatrix}
2 & 3 & -1 \\
4 & 0 & 7
\end{bmatrix}
+ \begin{bmatrix}
-1 & 2 & 1 \\
5 & -3 & 4
\end{bmatrix}
= \begin{bmatrix}
2 + (-1) & 3 + 2 & -1 + 1 \\
4 + 5 & 0 + (-3) & 7 + 4
\end{bmatrix}

= \begin{bmatrix}
-1 + 2 & 2 + 3 & 1 + (-1) \\
5 + 4 & -3 + 0 & 4 + 7
\end{bmatrix}

= \begin{bmatrix}
-1 & 2 & 1 \\
5 & -3 & 4
\end{bmatrix}
+ \begin{bmatrix}
2 & 3 & -1 \\
4 & 0 & 7
\end{bmatrix}
\]
The Zero Matrix

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \text{ 2 by 2 square zero matrix}
\end{equation}

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \text{ 2 by 3 zero matrix}
\end{equation}

\[
\begin{bmatrix}
0 & 0 & 0 \\
\end{bmatrix}, \text{ 1 by 3 zero matrix}
\end{equation}

\[
A + 0 = 0 + A = A
\]
OBJECTIVE 2

2 Find Scalar Multiples of a Matrix
EXAMPLE

Operations Using Matrices

\[ A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & -4 \end{bmatrix} \]

Find:  
(a) \(4A\)  
(b) \(\frac{1}{3}C\)  
(c) \(3A - 2B\)
Properties of Scalar Multiplication

\[ k(hA) = (kh)A \]
\[ (k + h)A = kA + hA \]
\[ k(A + B) = kA + kB \]
OBJECTIVE 3

3 Find the Product of Two Matrices
A row vector $R$ is a 1 by $n$ matrix

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

A column vector $C$ is an $n$ by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The product $RC$ of $R$ times $C$ is defined as the number

$$RC = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1 c_1 + r_2 c_2 + \cdots + r_n c_n$$
EXAMPLE

The Product of a Row Vector by a Column Vector

Find $RC$ if $R = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$
A clothing store sells ladies blouses for $40, scarves for $20, and wool suits for $300. Last month, the store had sales consisting of 200 blouses, 100 scarves, and 10 suits. What was the total revenue due to these sales?
Let $A$ denote an $m$ by $r$ matrix, and let $B$ denote an $r$ by $n$ matrix. The **product** $AB$ is defined as the $m$ by $n$ matrix whose entry in row $i$, column $j$ is the product of the $i$th row of $A$ and the $j$th column of $B$. 

$A$

$m$ by $r$

$B$

$r$ by $n$

Must be same for $AB$ to be defined

$AB$ is $m$ by $n$. 
EXAMPLE

Multiplying Two Matrices

Find the product $AB$ if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$
EXAMPLE  Multiplying Two Matrices

Find the product $BA$ if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Recall from last example:

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix}$$
**EXAMPLE**

Multiplying Two Square Matrices

\[
A = \begin{bmatrix}
1 & 3 \\
-2 & -7
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
-2 & 0 \\
3 & 4
\end{bmatrix}
\]

find: \( (a) \ AB \quad (b) \ BA \)
Theorem

Matrix multiplication is not commutative.

Associative Property of Matrix Multiplication

\[ A(BC) = (AB)C \]

Distributive Property

\[ A(B + C) = AB + AC \]
For an $n$ by $n$ square matrix, the entries located in row $i$, column $i$, $1 \leq i \leq n$, are called the **diagonal entries**. An $n$ by $n$ square matrix whose diagonal entries are 1’s, while all other entries are 0’s, is called the **identity matrix** $I_n$. For example,

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and so on.
EXAMPLE

Multiplication with an Identity Matrix

\[ A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix} \]

Find: \( (a) \ AI_3 \) \hspace{1cm} (b) \( I_2A \) \hspace{1cm} (c) \( BI_2 \)
Identity Property

If $A$ is an $m$ by $n$ matrix, then

$$I_mA = A \quad \text{and} \quad AI_n = A$$

If $A$ is an $n$ by $n$ square matrix, then

$$AI_n = I_nA = A$$
OBJECTIVE 4

4  Find the Inverse of a Matrix
DEFINITION

Let $A$ be a square $n$ by $n$ matrix. If there exists an $n$ by $n$ matrix $A^{-1}$, read “$A$ inverse,” for which

$$AA^{-1} = A^{-1}A = I_n$$

then $A^{-1}$ is called the inverse of the matrix $A$. 
Show that the inverse of \( A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} \) is \( A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \).
Procedure for Finding the Inverse of a Nonsingular Matrix

To find the inverse of an $n$ by $n$ nonsingular matrix $A$, proceed as follows:

**Step 1:** Form the matrix $[A | I_n]$.

**Step 2:** Transform the matrix $[A | I_n]$ into reduced row echelon form.

**Step 3:** The reduced row echelon form of $[A | I_n]$ will contain the identity matrix $I_n$ on the left of the vertical bar; the $n$ by $n$ matrix on the right of the vertical bar is the inverse of $A$. 
The matrix \( A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \) is nonsingular. Find its inverse.
EXAMPLE

Showing That a Matrix Has No Inverse

Show that the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has no inverse.
OBJECTIVE 5

Solve a System of Linear Equations Using an Inverse Matrix
Using the Inverse Matrix to Solve a System of Linear Equations

Solve the system of equations:

\[
\begin{align*}
  x - y + 2z &= 1 \\
  -y + 3z &= -2 \\
  2x + 2y + z &= -1
\end{align*}
\]