Section 12.4

Matrix Algebra

A **matrix** is defined as a rectangular array of numbers:

$$
\begin{array}{r}
\text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m
\end{array}
\overset{\begin{array}{cccccc} \text{Column 1} & \text{Column 2} & & \text{Column } j & & \text{Column } n \end{array}}{
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}}
$$

EXAMPLE Arranging Data in a Matrix

In a survey of 900 people, the following information was obtained:

200 males	Thought federal defense spending was too high
150 males	Thought federal defense spending was too low
45 males	Had no opinion
315 females	Thought federal defense spending was too high
125 females	Thought federal defense spending was too low
65 females	Had no opinion

We can arrange these data in a rectangular array as follows:

	Too High	Too Low	No Opinion
Male	200	150	45
Female	315	125	65

or as the matrix

$$\begin{bmatrix} 200 & 150 & 45 \\ 315 & 125 & 65 \end{bmatrix}$$

This matrix has two rows (representing males and females) and three columns (representing "too high," "too low," and "no opinion").

EXAMPLE Examples of Matrices

(a) $\begin{bmatrix} 5 & 0 \\ -6 & 1 \end{bmatrix}$ A 2 by 2 square matrix

(b) $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$ A 1 by 3 matrix

(c) $\begin{bmatrix} 6 & -2 & 4 \\ 4 & 3 & 5 \\ 8 & 0 & 1 \end{bmatrix}$ A 3 by 3 square matrix

OBJECTIVE 1

1 Find the Sum and Difference of Two Matrices

DEFINITION

Two m by n matrices A and B are said to be **equal**, written as

$$A = B$$

provided that each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B.

EXAMPLE Adding and Subtracting Matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -4 \end{bmatrix}$$

Find: (a) $A + B$ (b) $A - B$

Commutative Property of Matrix Addition

$$A + B = B + A$$

Associative Property of Matrix Addition

$$(A + B) + C = A + (B + C)$$

EXAMPLE

Demonstrating the Commutative Property

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+2 & -1+1 \\ 4+5 & 0+(-3) & 7+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & 2+3 & 1+(-1) \\ 5+4 & -3+0 & 4+7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix}$$

The Zero Matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ 2 by 2 square zero matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ 2 by 3 zero matrix

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$ 1 by 3 zero matrix

$$A + 0 = 0 + A = A$$

OBJECTIVE 2

2 Find Scalar Multiples of a Matrix

EXAMPLE Operations Using Matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \\ 0 & -4 \end{bmatrix}$$

Find: (a) $4A$ (b) $\frac{1}{3}C$ (c) $3A - 2B$

Properties of Scalar Multiplication

$$k(hA) = (kh)A$$
$$(k + h)A = kA + hA$$
$$k(A + B) = kA + kB$$

OBJECTIVE 3

3 Find the Product of Two Matrices

A **row vector** ***R*** is a 1 by n matrix

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

A **column vector** ***C*** is an n by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The **product** ***RC*** of R times C is defined as the number

$$RC = \begin{bmatrix} r_1 & r_2 \cdots r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

EXAMPLE

The Product of a Row Vector by a Column Vector

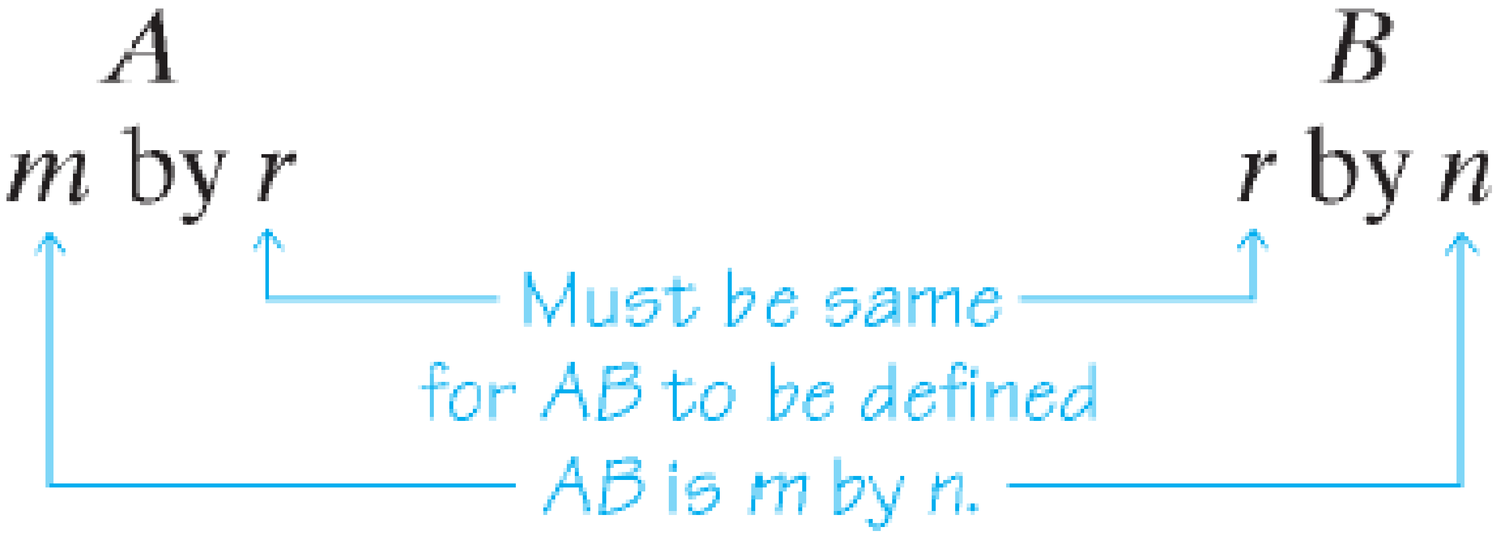
Find RC if $R = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

EXAMPLE

Using Matrices to Compute Revenue

A clothing store sells ladies blouses for $40, scarves for $20, and wool suits for $300. Last month, the store had sales consisting of 200 blouses, 100 scarves, and 10 suits. What was the total revenue due to these sales?

Let A denote an m by r matrix, and let B denote an r by n matrix. The **product** AB is defined as the m by n matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B.



EXAMPLE Multiplying Two Matrices

Find the product AB if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

EXAMPLE Multiplying Two Matrices

Find the product *BA* if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Recall from last example:

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix}$$

EXAMPLE **Multiplying Two Square Matrices**

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}$$

find: (a) AB (b) BA

Theorem

Matrix multiplication is not commutative.

Associative Property of Matrix Multiplication

$$A(BC) = (AB)C$$

Distributive Property

$$A(B + C) = AB + AC$$

For an n by n square matrix, the entries located in row i, column i, $1 \leq i \leq n$, are called the **diagonal entries**. An n by n square matrix whose diagonal entries are 1's, while all other entries are 0's, is called the **identity matrix** $\boldsymbol{I_n}$. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on.

EXAMPLE

Multiplication with an Identity Matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

Identity Property

If A is an m by n matrix, then

$$I_m A = A \quad \text{and} \quad A I_n = A$$

If A is an n by n square matrix, then

$$A I_n = I_n A = A$$

OBJECTIVE 4

4 Find the Inverse of a Matrix

DEFINITION

Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which

$$AA^{-1} = A^{-1}A = I_n$$

then A^{-1} is called the **inverse** of the matrix A.

EXAMPLE

Multiplying a Matrix by Its Inverse

Show that the inverse of $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ is $A^{-1} \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$

Procedure for Finding the Inverse of a Nonsingular Matrix

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

STEP 1: Form the matrix $[A|I_n]$.

STEP 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

STEP 3: The reduced row echelon form of $[A|I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

EXAMPLE Finding the Inverse of a Matrix

The matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ is nonsingular. Find its inverse.

EXAMPLE

Showing That a Matrix Has No Inverse

Show that the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has no inverse.

OBJECTIVE 5

5 Solve a System of Linear Equations Using an Inverse Matrix

EXAMPLE

Using the Inverse Matrix to Solve a System of Linear Equations

Solve the system of equations: $\begin{cases} x - y + 2z = 1 \\ \quad - y + 3z = -2 \\ 2x + 2y + z = -1 \end{cases}$