Section 4.3
Quadratic Functions and Their Properties
**Quadratic Functions**

**DEFINITION**

A **quadratic function** is a function of the form

\[ f(x) = ax^2 + bx + c \]

where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \). The domain of a quadratic function consists of all real numbers.
suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price \( p \) (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number \( x \) of calculators sold and the price \( p \) per calculator may be approximated by the linear equation

\[
x = 21,000 - 150p
\]

<table>
<thead>
<tr>
<th>Price per Calculator, ( p ) (Dollars)</th>
<th>Number of Calculators, ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>12,000</td>
</tr>
<tr>
<td>65</td>
<td>11,250</td>
</tr>
<tr>
<td>70</td>
<td>10,500</td>
</tr>
<tr>
<td>75</td>
<td>9,750</td>
</tr>
<tr>
<td>80</td>
<td>9,000</td>
</tr>
<tr>
<td>85</td>
<td>8,250</td>
</tr>
<tr>
<td>90</td>
<td>7,500</td>
</tr>
</tbody>
</table>

Then the revenue \( R \) derived from selling \( x \) calculators at the price \( p \) per calculator is equal to the unit selling price \( p \) of the product times the number \( x \) of units actually sold.
A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton’s second law of motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.
OBJECTIVE 1

1. Graph a Quadratic Function Using Transformations
\[ f(x) = ax^2, \ a > 0, \text{ for } a = 1, \ a = \frac{1}{2}, \text{ and } a = 3. \]
$f(x) = ax^2$ for $a < 0$. 
Graphs of a quadratic function, 
\[ f(x) = ax^2 + bx + c, \quad a \neq 0 \]

- **Axis of symmetry**
- **Vertex is highest point**

(a) **Opens up**
- Vertex is lowest point
- \( a > 0 \)

(b) **Opens down**
- Axis of symmetry
- \( a < 0 \)
Graphing a Quadratic Function Using Transformations

Graph the function $f(x) = -2x^2 + 6x + 2$

Find the vertex and axis of symmetry.
If \( h = -\frac{b}{2a} \) and \( k = \frac{4ac - b^2}{4a} \), then

\[
f(x) = ax^2 + bx + c = a(x - h)^2 + k
\]
OBJECTIVE 2

Identify the Vertex and Axis of Symmetry of a Quadratic Function
Properties of the Graph of a Quadratic Function

\[ f(x) = ax^2 + bx + c \quad a \neq 0 \]

Vertex \( = \left( -\frac{b}{2a}, f\left(-\frac{b}{2a}\right) \right) \)  
Axis of symmetry: the line \( x = -\frac{b}{2a} \)

Parabola opens up if \( a > 0 \); the vertex is a minimum point.
Parabola opens down if \( a < 0 \); the vertex is a maximum point.
Without graphing, locate the vertex and axis of symmetry of the parabola defined by \( f(x) = 3x^2 + 12x - 5 \). Does it open up or down?

\[
\text{Vertex} = \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right)
\]
OBJECTIVE 3

3. Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts
The $x$-Intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct $x$-intercepts so it crosses the $x$-axis in two places.

2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one $x$-intercept so it touches the $x$-axis at its vertex.

3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no $x$-intercept so it does not cross or touch the $x$-axis.
\[ f(x) = ax^2 + bx + c, \ a > 0 \]

(a) \[ b^2 - 4ac > 0 \]
Two \( x \)-intercepts

(b) \[ b^2 - 4ac = 0 \]
One \( x \)-intercept

(c) \[ b^2 - 4ac < 0 \]
No \( x \)-intercepts
Graph \( f(x) = 3x^2 + 12x - 5 \) using its properties.
Determine the domain and the range of \( f \).
Determine where \( f \) is increasing and where it is decreasing.
Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $x^2 + 4x + 4$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, $y$-intercept, and $x$-intercepts, if any.
(b) Determine the domain and the range of $f$.
(c) Determine where $f$ is increasing and where it is decreasing
(a) Graph \(- x^2 + 4x + 7\) by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, \(y\)-intercept, and \(x\)-intercepts, if any.

(b) Determine the domain and the range of \(f\).

(c) Determine where \(f\) is increasing and where it is decreasing.
**SUMMARY**  Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$, by Hand

**Option 1**

**Step 1:** Complete the square in $x$ to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

**Step 2:** Graph the function in stages using transformations.

**Option 2**

**Step 1:** Determine whether the graph of $f$ opens up or down.

**Step 2:** Determine the vertex \( \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \) and the axis of symmetry, $x = -\frac{b}{2a}$.

**Step 3:** Determine the $y$-intercept, $f(0)$. Determine the $x$-intercept(s), if any.

(a) If $b^2 - 4ac > 0$, then the graph of the quadratic function has two $x$-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.

(b) If $b^2 - 4ac = 0$, the vertex is the $x$-intercept.

(c) If $b^2 - 4ac < 0$, there are no $x$-intercepts.

**Step 4:** Determine an additional point by using the $y$-intercept and the axis of symmetry. Plot the points and draw the graph.
Given the vertex \((h, k)\) and one additional point on the graph of a quadratic function \(f(x) = ax^2 + bx + c, a \neq 0\), we can use

\[
\begin{align*}
  f(x) &= a(x - h)^2 + k
\end{align*}
\]

(3)

to obtain the quadratic function.
EXAMPLE

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is (-2, -5) and whose y intercept is – 1.
OBJECTIVE 4

4) Find the Maximum or Minimum Value of a Quadratic Function
Determine whether the quadratic function
\[ f(x) = -x^2 + 4x + 5 \]
has a maximum or minimum value. Then find the maximum or minimum value.