Section 4.3

Quadratic Functions and Their Properties

Quadratic Functions

DEFINITION

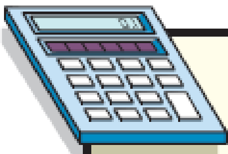
A **quadratic function** is a function of the form

$$\boxed{f(x) = ax^2 + bx + c}$$

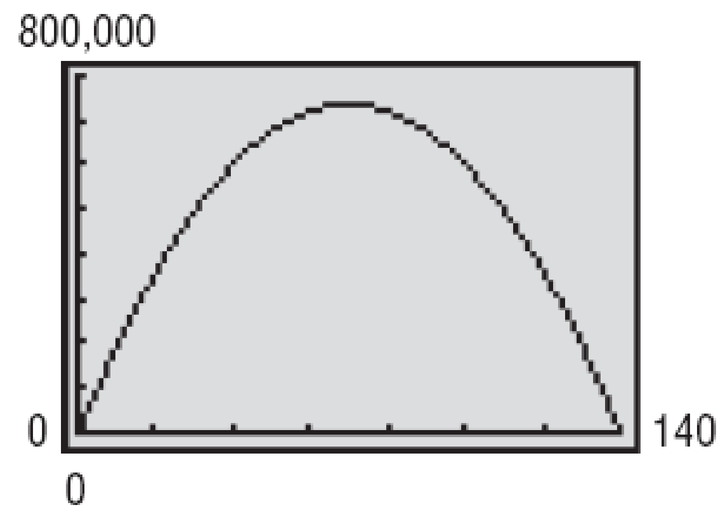
where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function consists of all real numbers.

suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price p (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number x of calculators sold and the price p per calculator may be approximated by the linear equation

$$x = 21{,}000 - 150p$$

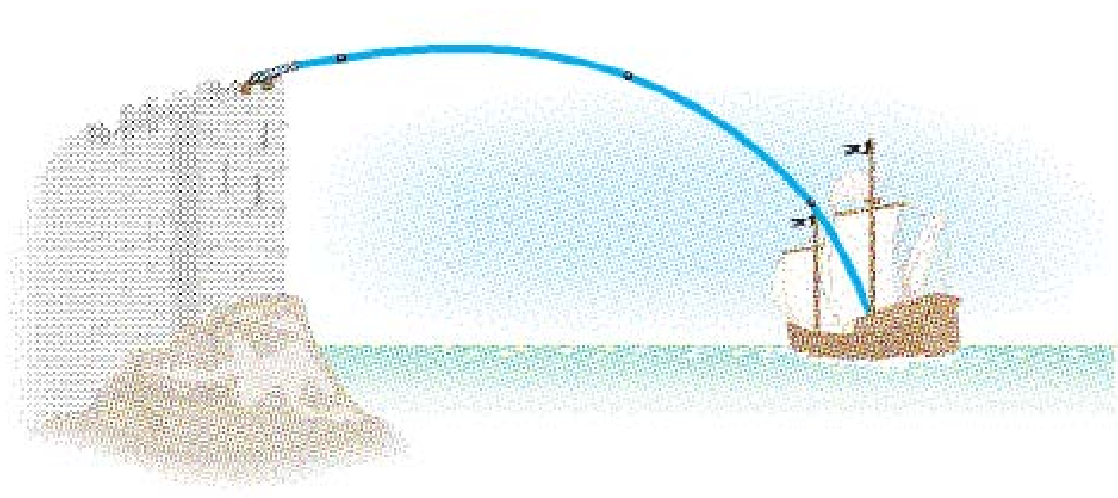


Price per Calculator, p (Dollars)	Number of Calculators, x
60	12,000
65	11,250
70	10,500
75	9,750
80	9,000
85	8,250
90	7,500



Then the revenue R derived from selling x calculators at the price p per calculator is equal to the unit selling price p of the product times the number x of units actually sold.

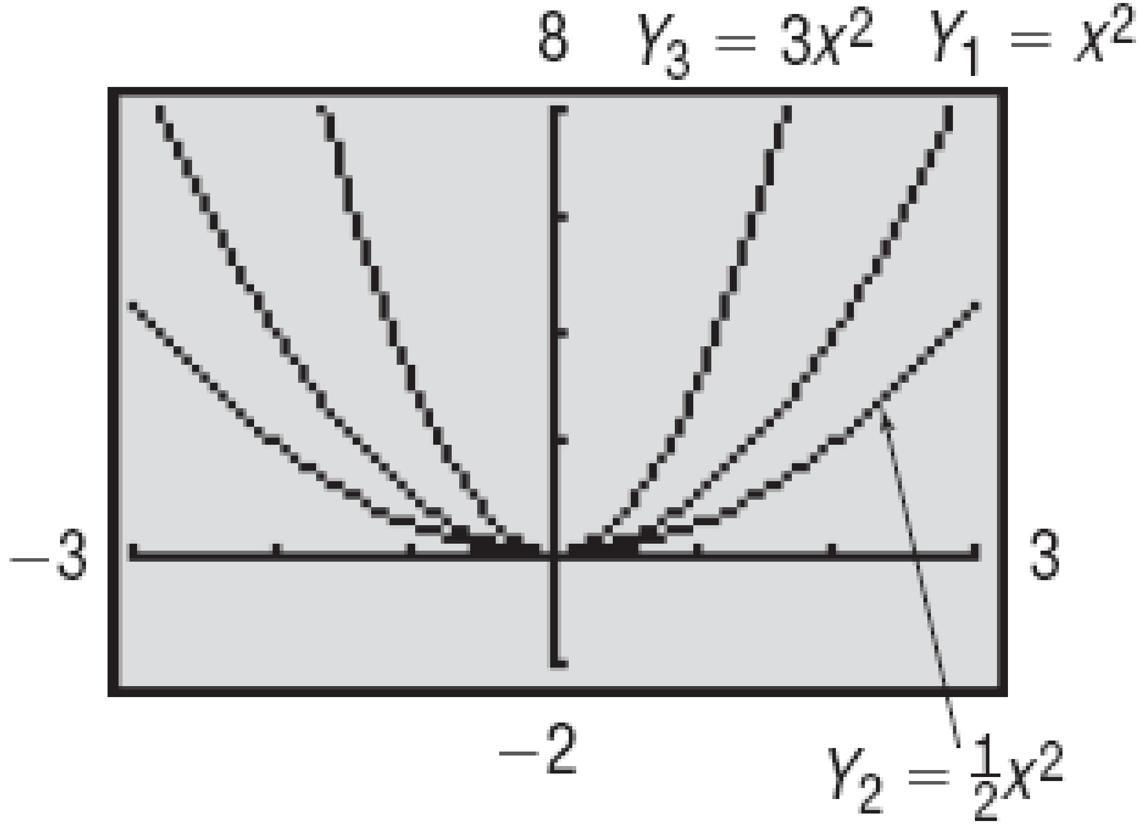
A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's second law of motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.



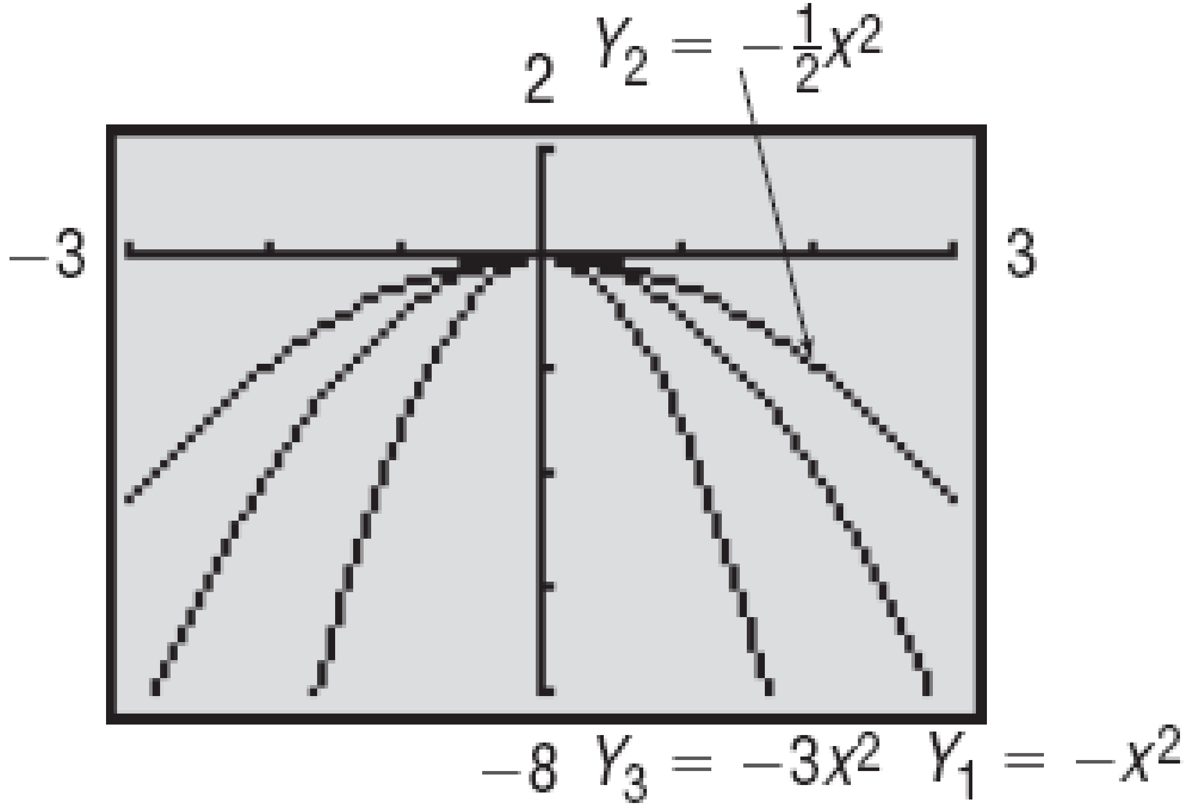
Path of a cannonball

OBJECTIVE 1

1 Graph a Quadratic Function Using Transformations

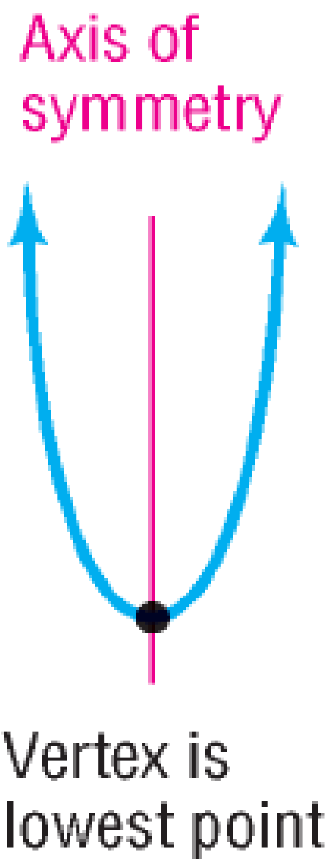


$f(x) = ax^2, a > 0,$ for $a = 1, a = \dfrac{1}{2},$ and $a = 3.$



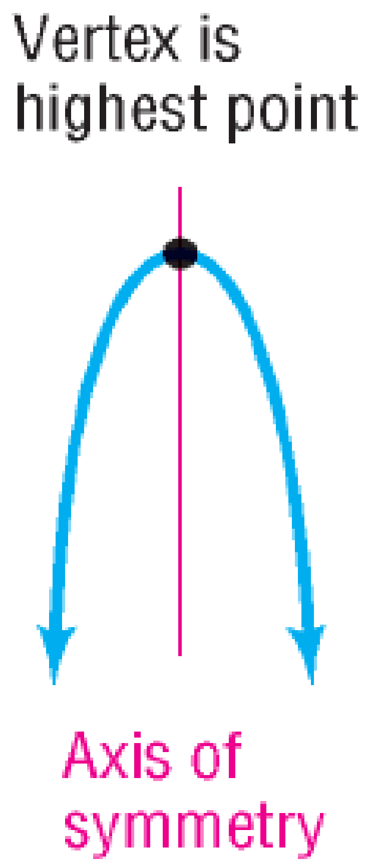
$$f(x) = ax^2 \text{ for } a < 0.$$

Graphs of a quadratic function,
$f(x) = ax^2 + bx + c, a \neq 0$



(a) Opens up

$a > 0$



(b) Opens down

$a < 0$

EXAMPLE

Graphing a Quadratic Function Using Transformations

Graph the function $f\left(x\right)=-2x^{2}+6x+2$

Find the vertex and axis of symmetry.

If $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$, then

$$\boxed{f(x) = ax^2 + bx + c = a(x - h)^2 + k}$$

OBJECTIVE 2

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

Properties of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c \qquad a \neq 0$$

Vertex $= \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$ Axis of symmetry: the line $x = -\dfrac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point.

EXAMPLE

Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = 3x^2 + 12x - 5$. Does it open up or down?

$$\boxed{\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)}$$

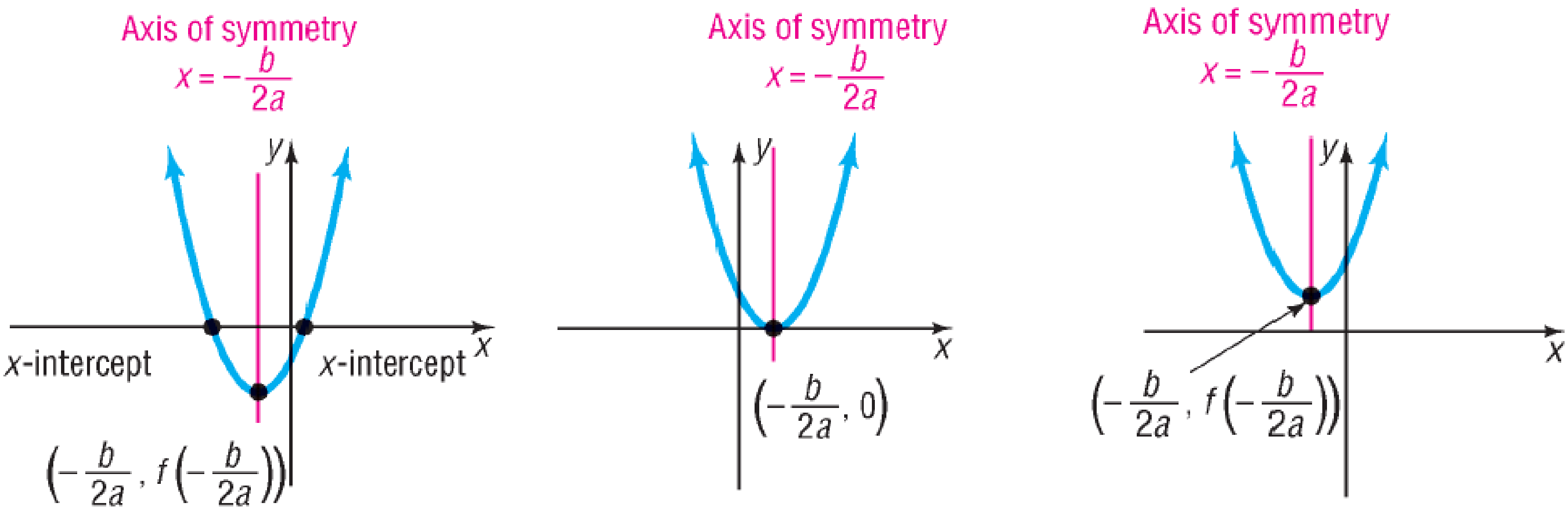
OBJECTIVE 3

3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The *x*-Intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts so it crosses the x-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept so it touches the x-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept so it does not cross or touch the x-axis.

$$f(x) = ax^2 + bx + c, a > 0$$



(a) $b^2 - 4ac > 0$
Two x-intercepts

(b) $b^2 - 4ac = 0$
One x-intercept

(c) $b^2 - 4ac < 0$
No x-intercepts

EXAMPLE How to Graph a Quadratic Function by Hand Using Its Properties

Graph $f(x) = 3x^2 + 12x - 5$ using its properties.

Determine the domain and the range of f.

Determine where f is increasing and where is is decreasing.

EXAMPLE

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $x^2 + 4x + 4$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
(b) Determine the domain and the range of f.
(c) Determine where f is increasing and where it is decreasing

EXAMPLE

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $-x^2+4x+7$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
(b) Determine the domain and the range of f.
(c) Determine where f is increasing and where it is decreasing.

SUMMARY Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$, by Hand

Option 1

Step 1: Complete the square in x to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

Step 2: Graph the function in stages using transformations.

Option 2

Step 1: Determine whether the graph of f opens up or down.

Step 2: Determine the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ and the axis of symmetry, $x = -\frac{b}{2a}$.

Step 3: Determine the y-intercept, $f(0)$. Determine the x-intercept(s), if any.

(a) If $b^2 - 4ac > 0$, then the graph of the quadratic function has two x-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.

(b) If $b^2 - 4ac = 0$, the vertex is the x-intercept.

(c) If $b^2 - 4ac < 0$, there are no x-intercepts.

Step 4: Determine an additional point by using the y-intercept and the axis of symmetry. Plot the points and draw the graph.

Given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, we can use

$$\boxed{f(x) = a(x - h)^2 + k \qquad \textbf{(3)}}$$

to obtain the quadratic function.

EXAMPLE

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is (-2, -5) and whose y intercept is – 1.

OBJECTIVE 4

4 Find the Maximum or Minimum Value of a Quadratic Function

EXAMPLE

Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

$$f(x) = -x^2 + 4x + 5$$

has a maximum or minimum value.

Then find the maximum or minimum value.

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$