Section 5.1

Polynomial Functions and Models

OBJECTIVE 1

1 Identify Polynomial Functions and Their Degree

DEFINITION

A **polynomial function** is a function of the form

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers.

EXAMPLE

Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 3x^5 - 4x^4 + 2x^3 + 5$

(b) $g(x) = 3x^2 + 5x - 10$

(c) $h(x) = 3x^{\frac{1}{3}} - 5$

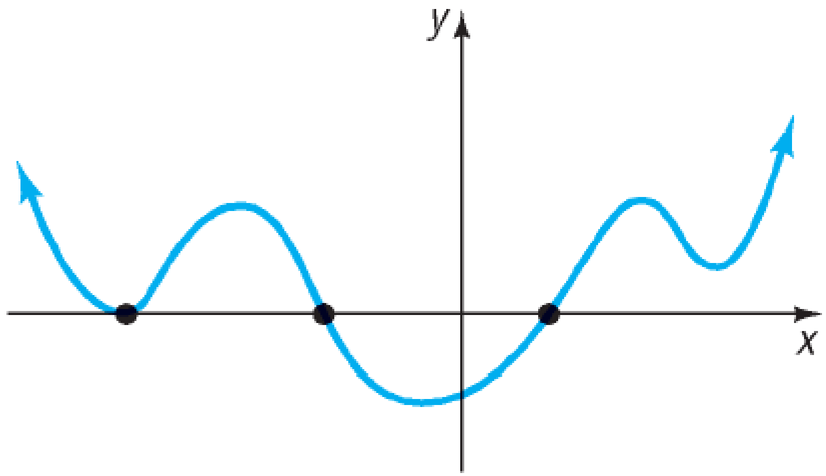
(d) $F(x) = 2x^{-3} + 3x - 8$

(e) $G(x) = -5$

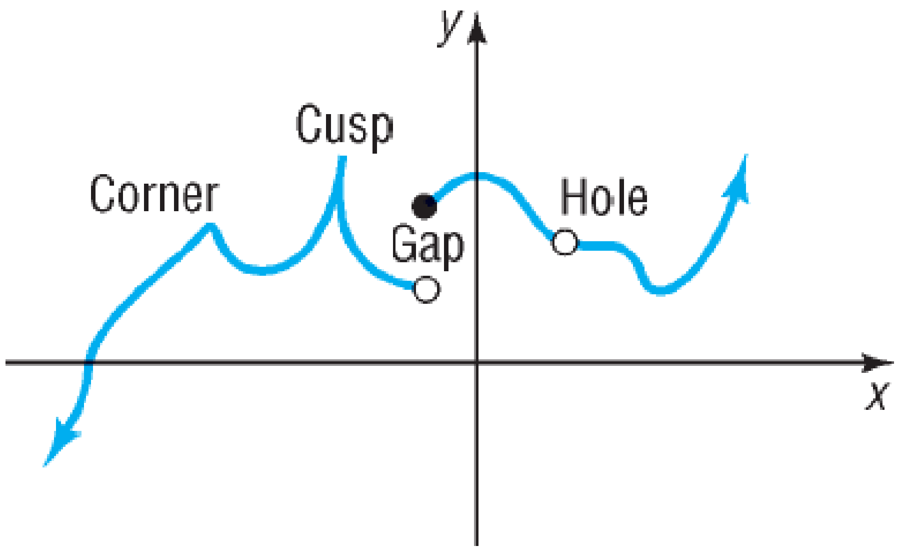
(f) $H(s) = 3s(2s^2 - 1)$

Summary of the Properties of the Graphs of Polynomial Functions

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: Graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$



(a) Graph of a polynomial function: smooth, continuous



(b) Cannot be the graph of a polynomial function

Power Functions

DEFINITION

A **power function of degree n** is a monomial of the form

$$\boxed{f(x) = ax^n \qquad \textbf{(2)}}$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

$f(x) = 3x$ degree 1

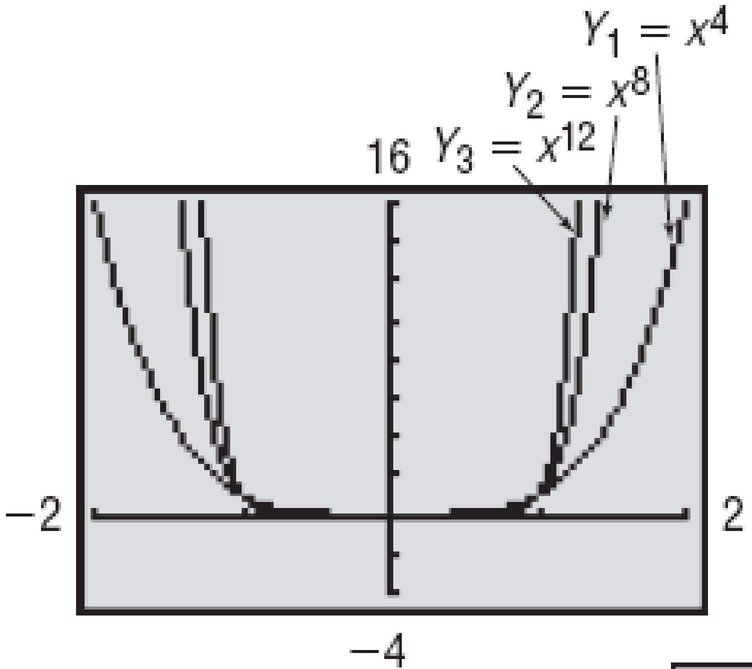
$f(x) = -5x^2$ degree 2

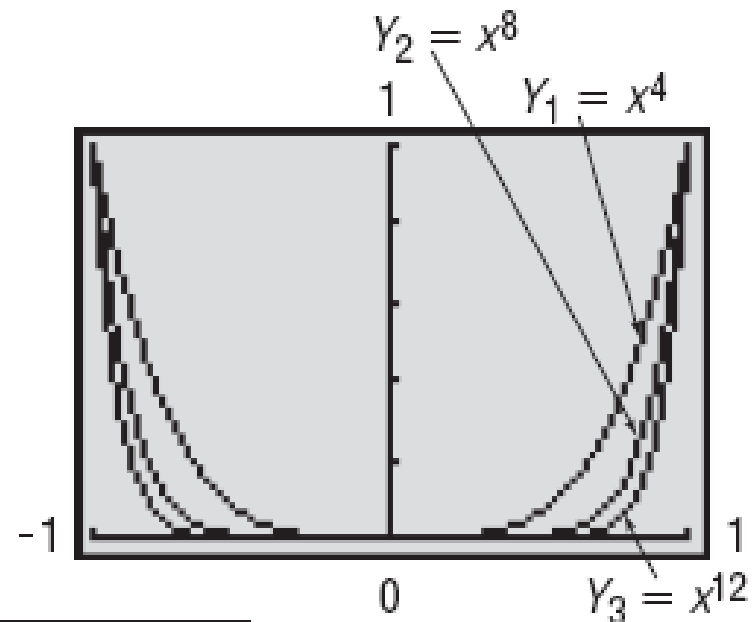
$f(x) = 8x^3$ degree 3

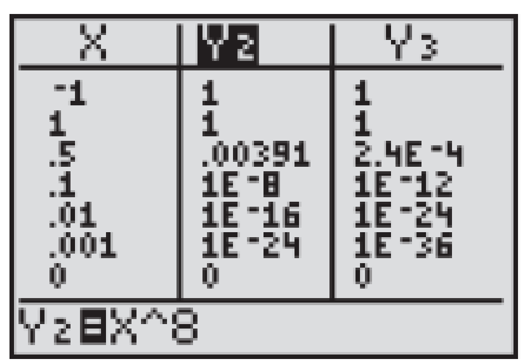
$f(x) = -5x^4$ degree 4

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-4 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^4$. On the same screen, graph $Y_2 = g(x) = x^8$. Now, also on the same screen, graph $Y_3 = h(x) = x^{12}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $0 \leq y \leq 1$. What do you notice?





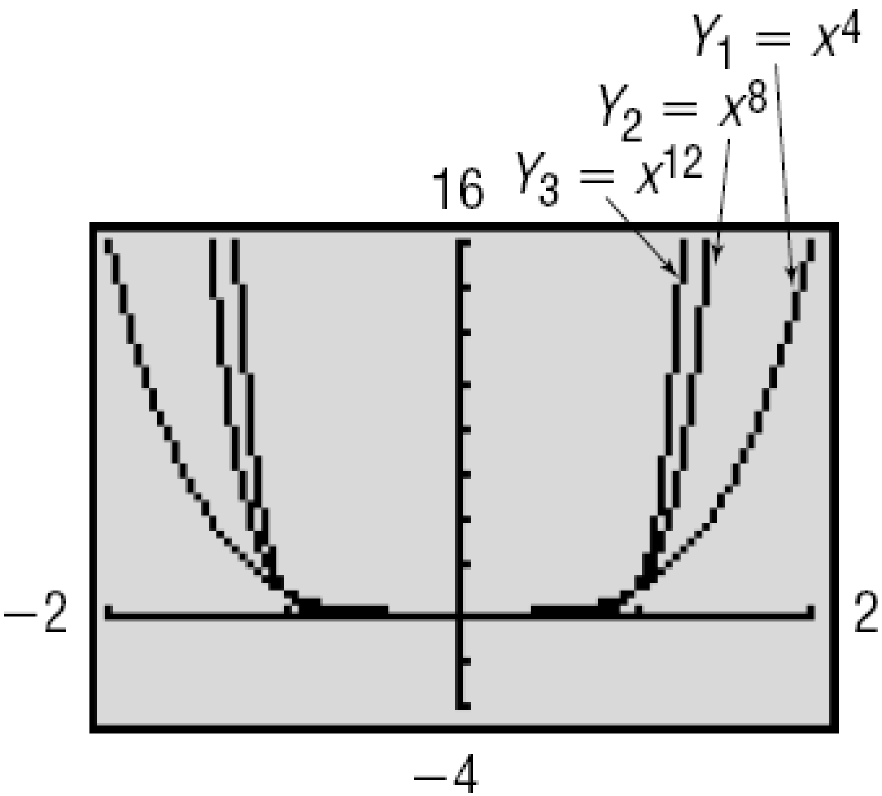


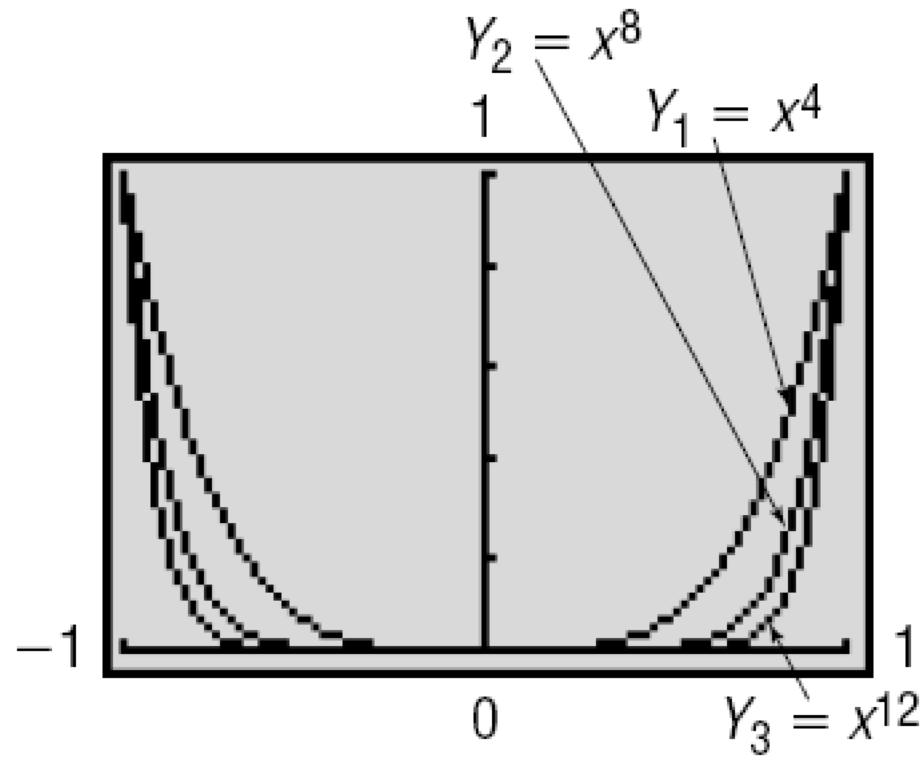
X	Y2	Y3
-1	1	1
1	1	1
.5	.00391	2.4E-4
.1	1E-8	1E-12
.01	1E-16	1E-24
.001	1E-24	1E-36
0	0	0

Y2=X^8

Seeing the Concept

Graph $Y_1 = x^4$, $Y_2 = x^8$, and $Y_3 = x^{12}$ using the viewing rectangle $-2 \leq x \leq 2$, $-4 \leq y \leq 16$. Then graph each again using the viewing rectangle $-1 \leq x \leq 1$, $0 \leq y \leq 1$. See Figure 3. TRACE along one of the graphs to confirm that for x close to 0 the graph is above the x-axis and that for $x > 0$ the graph is increasing.



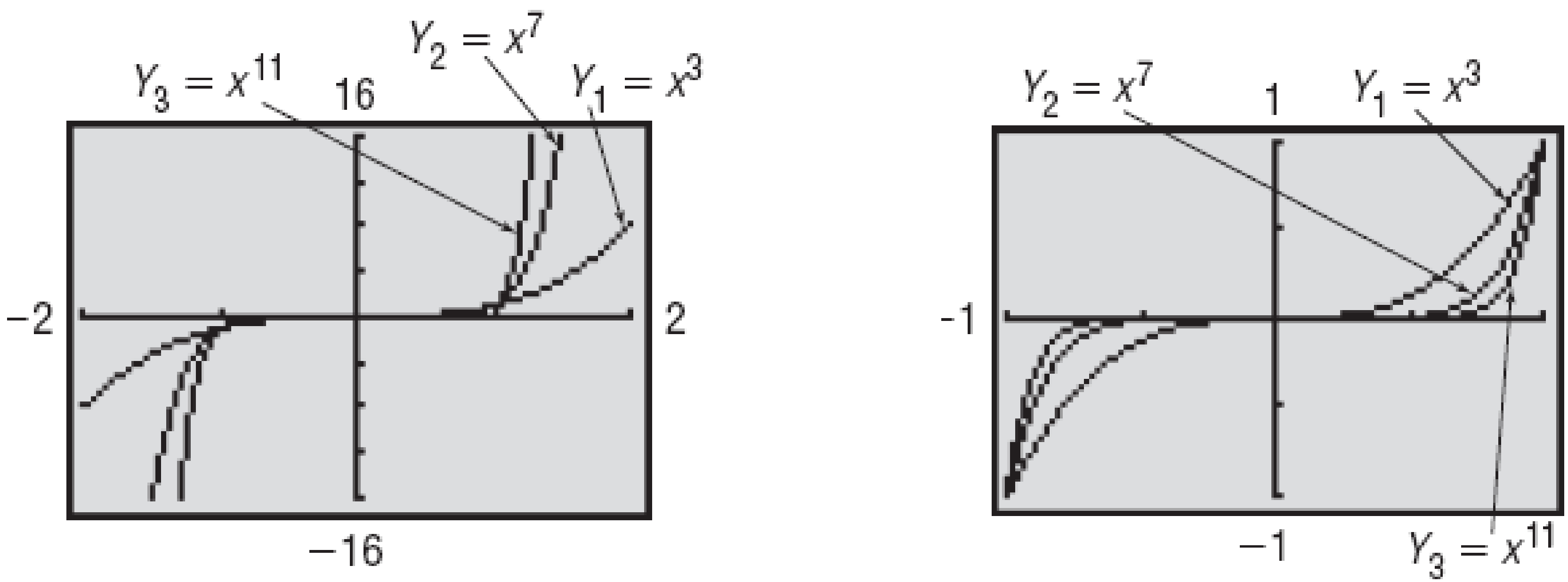


Properties of Power Functions, $f(x) = x^n$, n Is an Even Integer

1. f is an even function so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2, -16 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^3$. On the same screen, graph $Y_2 = g(x) = x^7$ and $Y_3 = h(x) = x^{11}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1, -1 \leq y \leq 1$. What do you notice?



Properties of Power Functions, $f(x) = x^n$, n Is an Odd Integer

1. f is an odd function so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

OBJECTIVE 2

2 Graph Polynomial Functions Using Transformations

EXAMPLE

Graphing Polynomial Functions Using Transformations

Graph: $2+x^5$

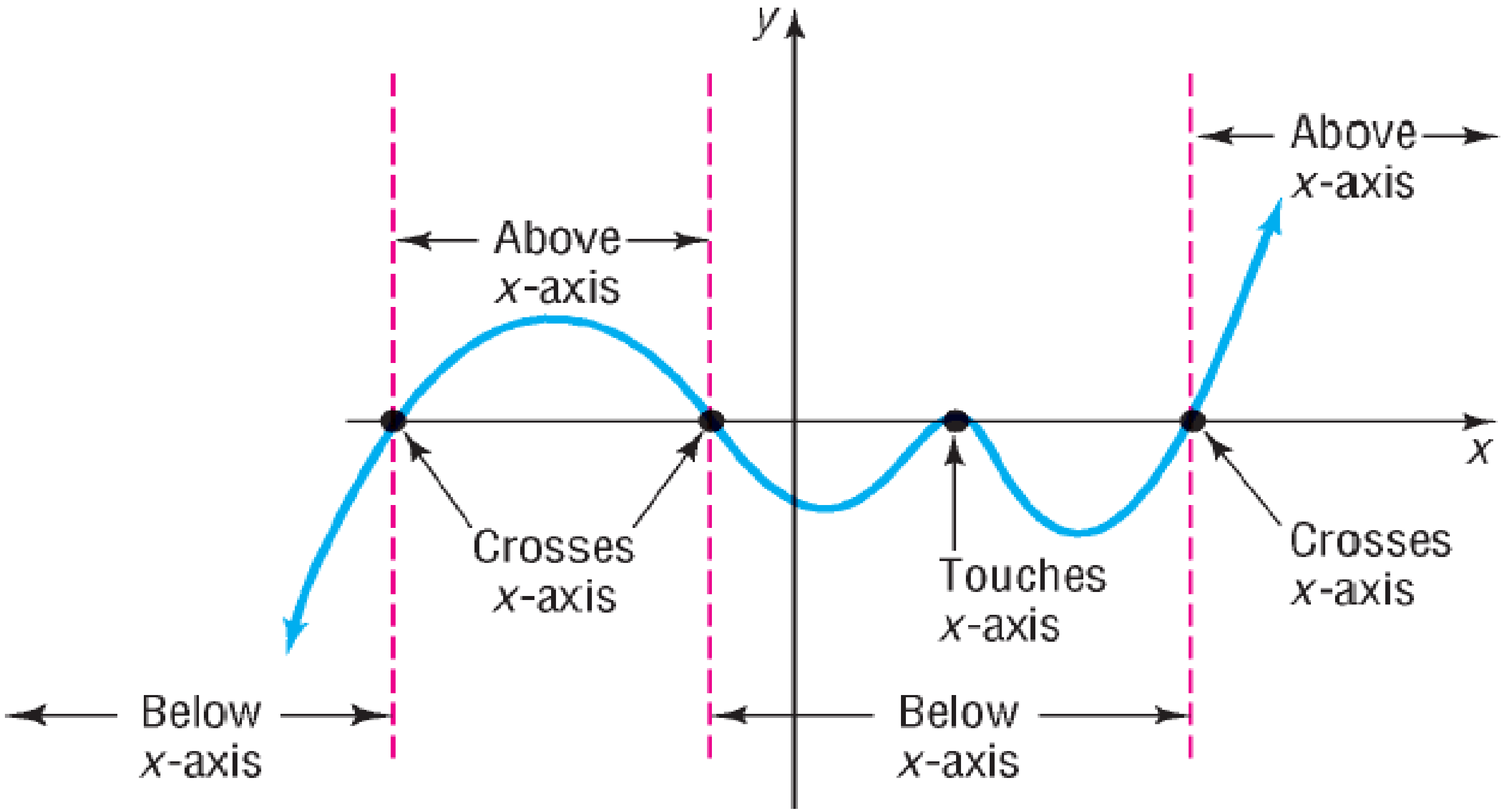
EXAMPLE

Graphing Polynomial Functions Using Transformations

Graph: $-2(x+1)^4$

OBJECTIVE 3

3 Identify the Real Zeros of a Polynomial Function and Their Multiplicity


y
x
Above
x-axis
Above
x-axis
Crosses
x-axis
Touches
x-axis
Crosses
x-axis
Below
x-axis
Below
x-axis

DEFINITION

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero of f**.

As a consequence of this definition, the following statements are equivalent.

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

EXAMPLE Finding a Polynomial from Its Zeros

Find a polynomial of degree 3 whose zeros are -4, -2, and 3.

DEFINITION

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**.

EXAMPLE

Identifying Zeros and Their Multiplicities

For the polynomial, list all zeros and their multiplicities.

$$f(x) = (x+4)^3 (x-2)^4$$

EXAMPLE Investigating the Role of Multiplicity

For the polynomial $f(x) = -x^3(x-3)^2(x+2)$

(a) Find the x- and y-intercepts of the graph of f.
(b) Using a graphing utility, graph the polynomial.
(c) For each x-intercept, determine whether it is of odd or even multiplicity.

If r Is a Zero of Even Multiplicity

Sign of $f(x)$ does not change from one side of r to the other side of r.

Graph **touches** x-axis at r.

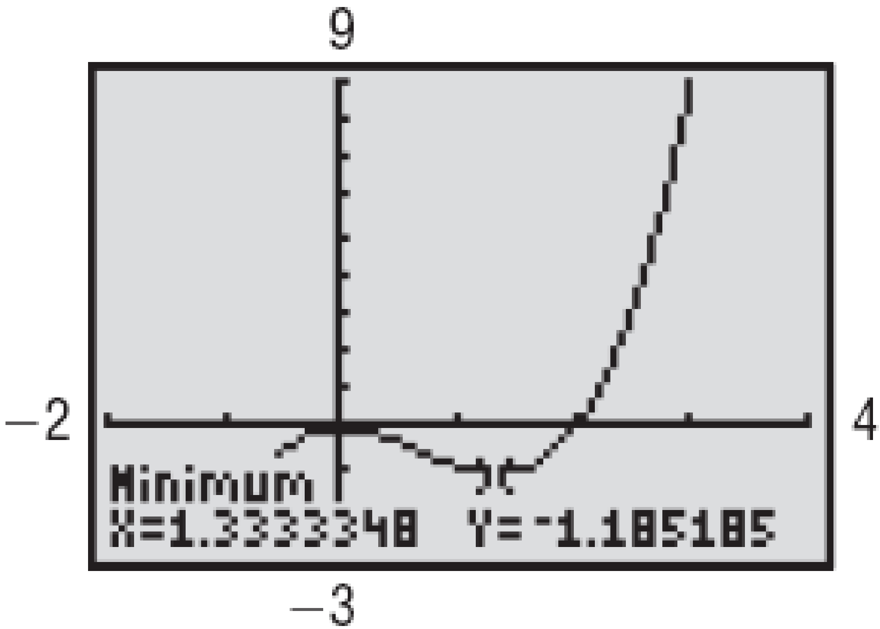
If r Is a Zero of Odd Multiplicity

Sign of $f(x)$ changes from one side of r to the other side of r.

Graph **crosses** x-axis at r.

Turning Points

$$f(x) = x^2(x - 2)$$



Exploration

Graph $Y_1 = x^3$, $Y_2 = x^3 - x$ and $Y_3 = x^3 + 3x^2 + 4$. How many turning points do you see? Graph $Y_1 = x^4$, $Y_2 = x^4 - \frac{4}{3}x^3$, and $Y_3 = x^4 - 2x^2$. How many turning points do you see? How does the number of turning points compare to the degree?

Turning Points

Theorem

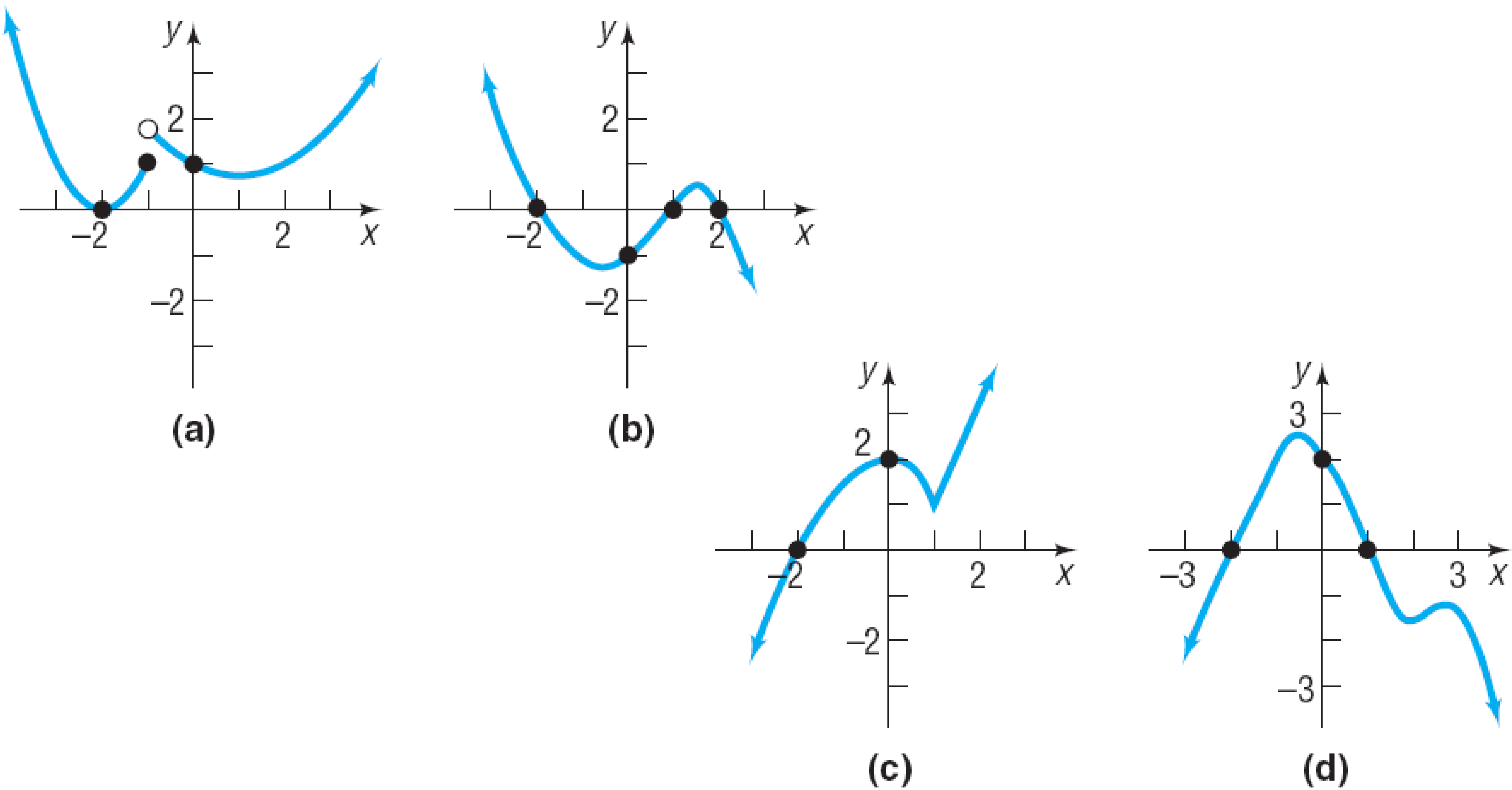
Turning Points

If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, the degree of f is at least n.

EXAMPLE Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 16 could be the graph of a polynomial function? For those that could, list the zeros and state the least degree the polynomial can have. For those that could not, say why not.



End Behavior

THEOREM

End Behavior

For large values of x, either positive or negative, the graph of the polynomial

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$$

resembles the graph of the power function

$$\boxed{y = a_n x^n}$$

Seeing the Concept

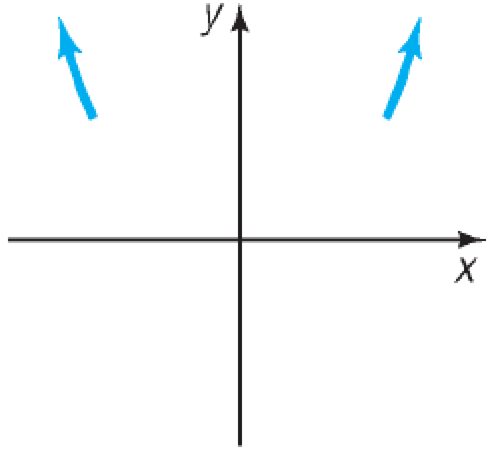
For each pair of functions Y_1 and Y_2 given in parts (a), (b), and (c), graph Y_1 and Y_2 on the same viewing window. Create a TABLE or TRACE for large positive and large negative values of x. What do you notice about the graphs of Y_1 and Y_2 as x becomes very large and positive or very large and negative?

(a) $Y_1 = x^2(x - 2)$; $Y_2 = x^3$

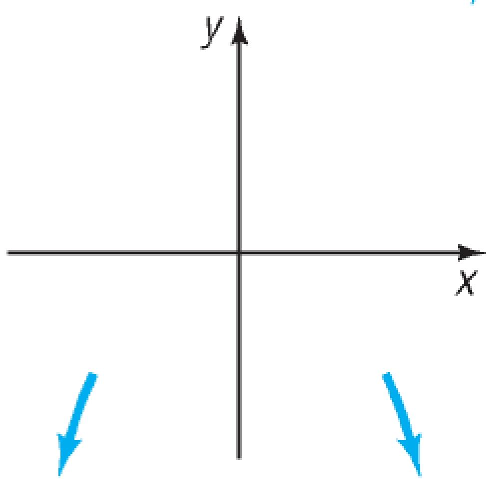
(b) $Y_1 = x^4 - 3x^3 + 7x - 3$; $Y_2 = x^4$

(c) $Y_1 = -2x^3 + 4x^2 - 8x + 10$; $Y_2 = -2x^3$

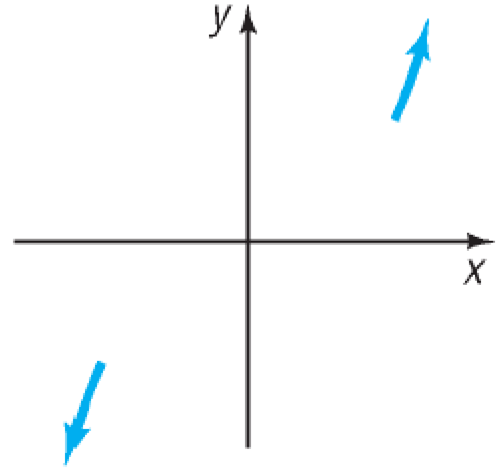
End Behavior $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$



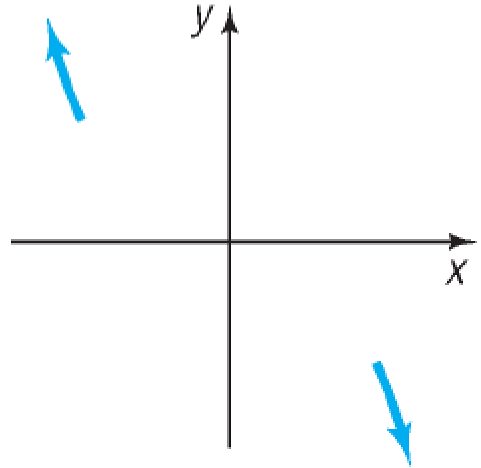
(a)
$n \geq 2$ even; $a_n > 0$



(b)
$n \geq 2$ even; $a_n < 0$



(c)
$n \geq 3$ odd; $a_n > 0$

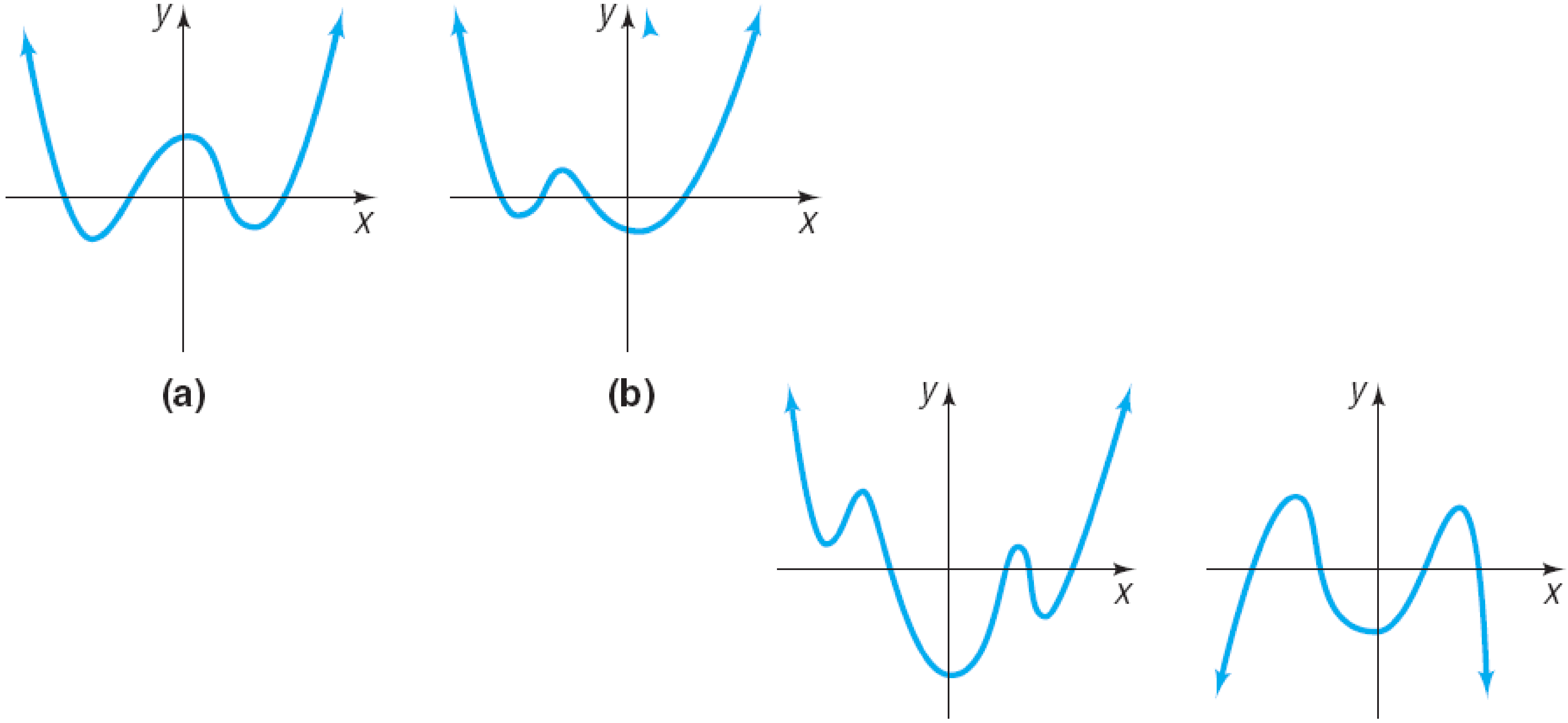


(d)
$n \geq 3$ odd; $a_n < 0$

EXAMPLE Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 18 could be the graph of

$$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6?$$



(a) (b) (c) (d)

Summary

Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$

Degree of the polynomial f: n

Maximum number of turning points: $n - 1$

At a zero of even multiplicity: The graph of f touches the x-axis.

At a zero of odd multiplicity: The graph of f crosses the x-axis.

Between zeros, the graph of f is either above or below the x-axis.

End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

OBJECTIVE 4

4 Analyze the Graph of a Polynomial Function

EXAMPLE

How to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function $f(x) = (x+4)^2(x+1)(x-2)$

Step-by-Step Solution

Step 1 Determine the end behavior of the graph of the function.

Step 2 Find the x- and y-intercepts of the graph of the function.

Step 3 Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4 Use a graphing utility to graph the function.

Step 5 Approximate the turning points of the graph.

Step 6 Use the information in Steps 1 to 5 to draw a complete graph of the function by hand.

Step 7 Find the domain and the range of the function.

Step 8 Use the graph to determine where the function is increasing and where it is decreasing.

SUMMARY Analyzing the Graph of a Polynomial Function

Step 1: Determine the end behavior of the graph of the function.

Step 2: Find the x- and y-intercepts of the graph of the function.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4: Use a graphing utility to graph the function.

Step 5: Approximate the turning points of the graph.

Step 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

Step 7: Find the domain and the range of the function.

Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

EXAMPLE

How to Use a Graphing Utility to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = -x^3 - 3.12x^2 + 3.512x - 5.1354$$

Step-by-Step Solution

STEP 1 Determine the end behavior of the graph of the function.

STEP 2 Graph the function using a graphing utility.

STEP 3 Use a graphing utility to approximate the x- and y-intercepts of the graph.

STEP 4 Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

STEP 5 Approximate the turning points of the graph.

STEP 6 Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

STEP 7 Find the domain and the range of the function.

STEP 8 Use the graph to determine where the function is increasing and where it is decreasing.

SUMMARY Using a Graphing Utility to Analyze the Graph of a Polynomial Function

Step 1: Determine the end behavior of the graph of the function.

Step 2: Graph the function using a graphing utility.

Step 3: Use a graphing utility to approximate the x- and y-intercepts of the graph.

Step 4: Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

Step 5: Approximate the turning points of the graph.

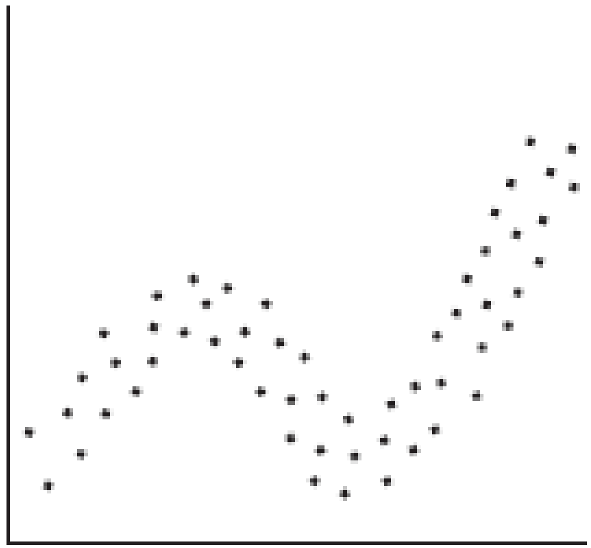
Step 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

Step 7: Find the domain and the range of the function.

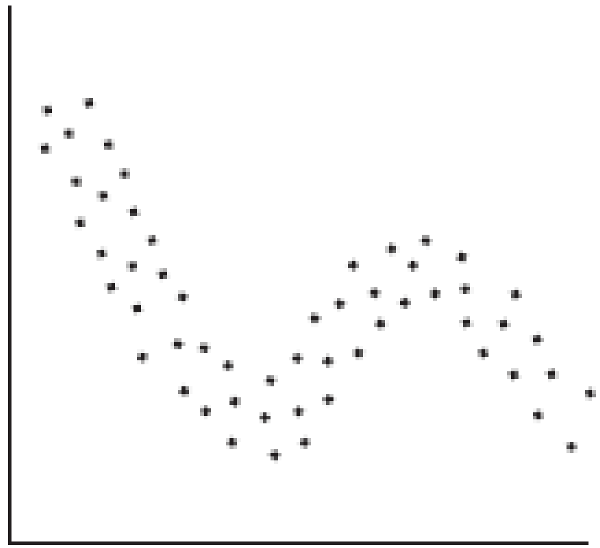
Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

OBJECTIVE 5

5 Build Cubic Models from Data



$y = ax^3 + bx^2 + cx + d, a > 0$



$y = ax^3 + bx^2 + cx + d, a < 0$

EXAMPLE

A Cubic Function of Best Fit

Number of Text Books, x	Cost, C
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

The data in Table 5 represent the weekly cost C (in thousands of dollars) of printing x thousand textbooks.

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables x and C.

(b) Using a graphing utility, find the cubic function of best fit $C = C(x)$ that models the relation between number of texts and cost.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the cost of printing 22 thousand texts per week.