Section 5.2

Properties of Rational Functions

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial.

The domain is the set of all real numbers except those for which the denominator q is 0.

OBJECTIVE 1

1 Find the Domain of a Rational Function

EXAMPLE **Finding the Domain of a Rational Function**

Find the domain of the following rational functions:

(a) $R(x)=\dfrac{x^2-4}{x+4}$

(b) $R(x)=\dfrac{x+6}{x^2+8x+12}$

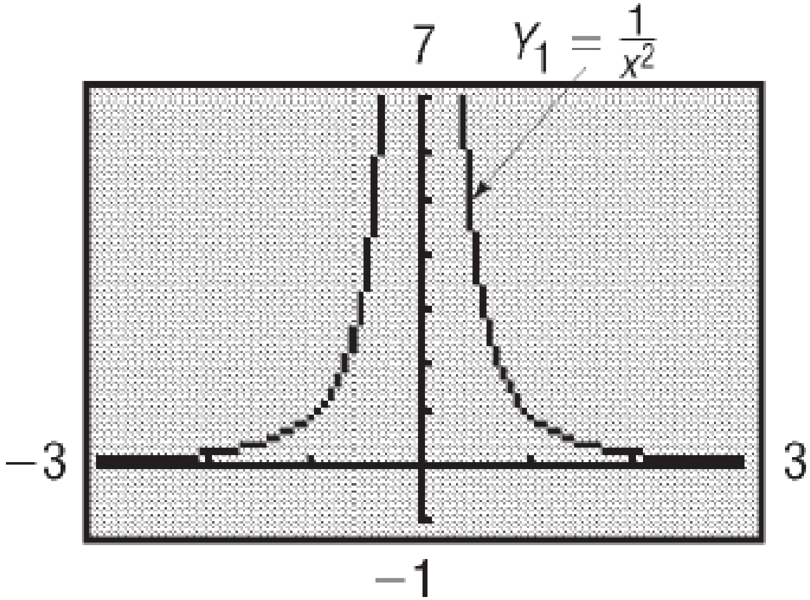
(c) $R(x)=\dfrac{x-5}{x^2+2}$

(d) $R(x)=\dfrac{x^2-9}{3}$

(e) $R(x)=\dfrac{x^2-4}{x+2}$

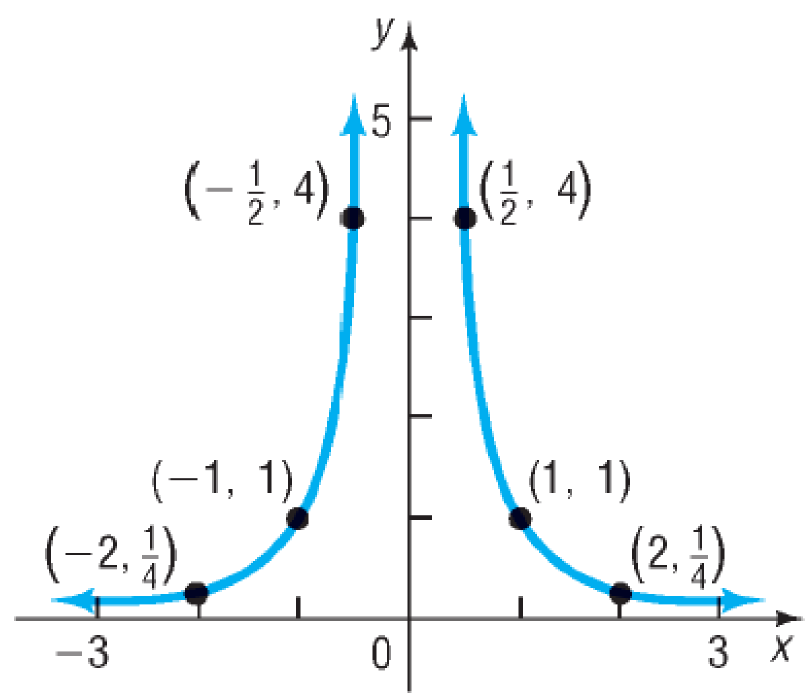
EXAMPLE Graphing $y = \frac{1}{x^2}$

Analyze the graph of $H(x) = \frac{1}{x^2}$.



X	Y1
.1	100
.01	10000
.001	1E6
1E-4	1E8
10	.01
100	1E-4
1000	1E-6

Y1 = 1/X²



EXAMPLE

Using Transformations to Graph a Rational Function

Graph the rational function: $R(x)=\dfrac{1}{(x-3)^2}+2$

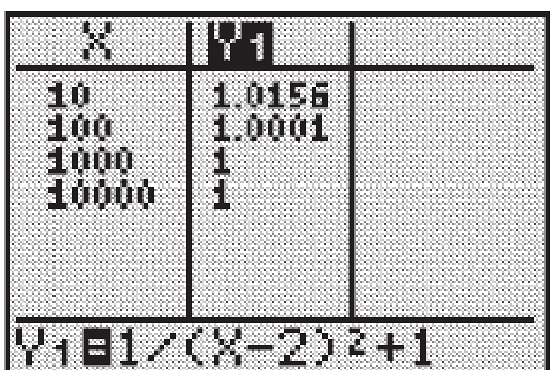
Asymptotes Exploration

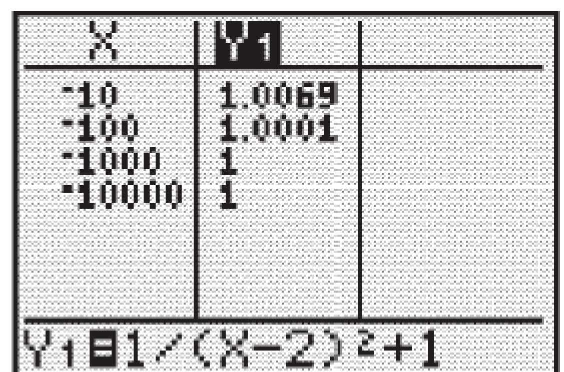
(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \frac{1}{(x-2)^2} + 1$ at $x = 10, 100, 1000,$ and $10{,}000$. What happens to the values of H as x becomes unbounded in the positive direction, symbolized by $\lim_{x \to \infty} H(x)$?

(b) Evaluate H at $x = -10, -100, -1000,$ and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, symbolized by $\lim_{x \to -\infty} H(x)$?

(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999,$ and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, symbolized by $\lim_{x \to 2^-} H(x)$?

(d) Evaluate H at $x = 2.5, 2.1, 2.01, 2.001,$ and 2.0001. What happens to the values of H as x approaches 2, $x > 2$, symbolized by $\lim_{x \to 2^+} H(x)$?





Asymptotes

Exploration

(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \dfrac{1}{(x-2)^2} + 1$ at $x = 10, 100, 1000,$ and $10{,}000$. What happens to the values of H as x becomes unbounded in the positive direction, symbolized by $\lim_{x \to \infty} H(x)$?

(b) Evaluate H at $x = -10, -100, -1000,$ and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, symbolized by $\lim_{x \to -\infty} H(x)$?

(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999,$ and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, symbolized by $\lim_{x \to 2^-} H(x)$?

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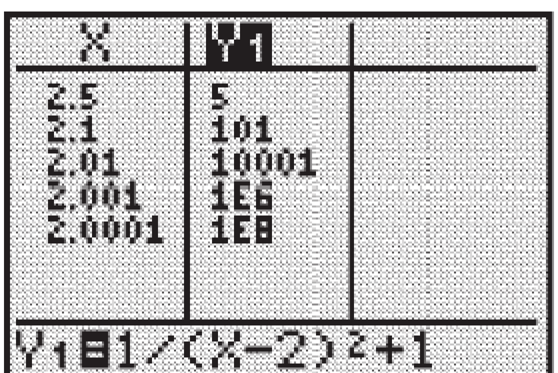
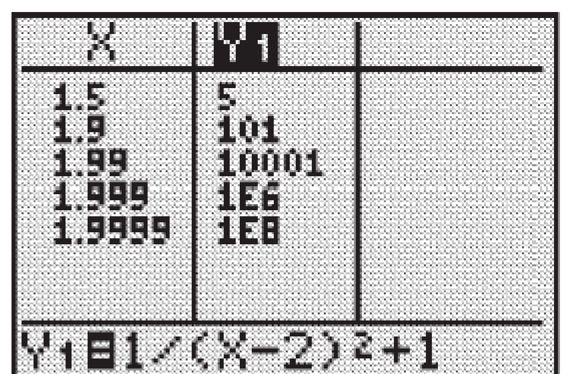
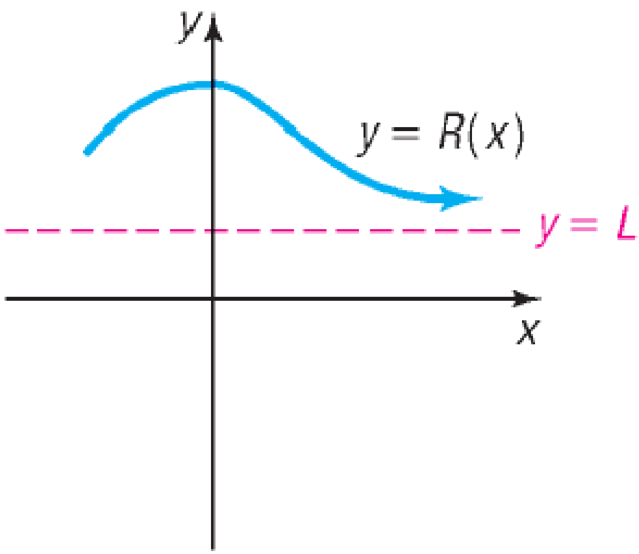
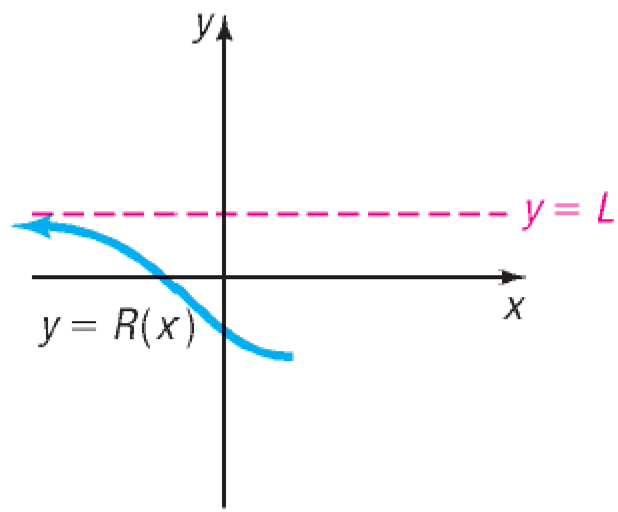




Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.



(a) End behavior:
As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x \to \infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.



(b) End behavior:
As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x \to -\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

Let R denote a function:
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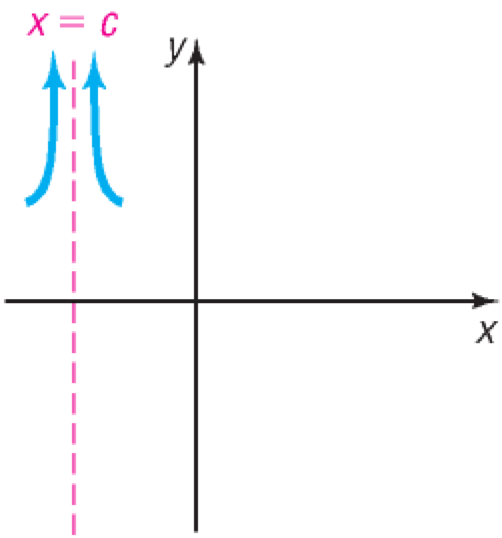
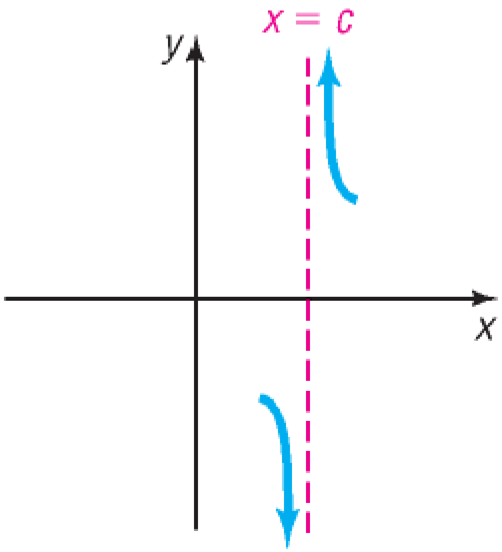
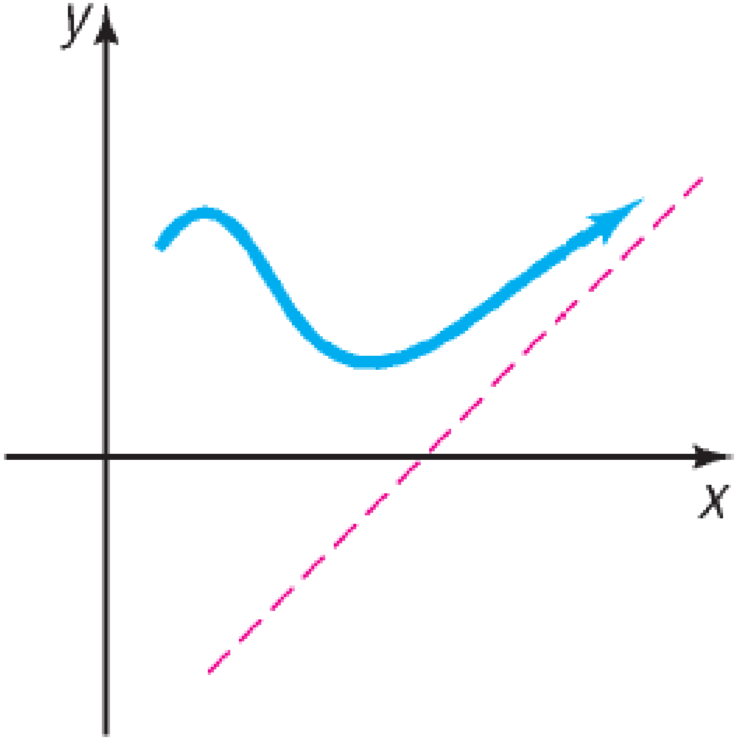


(c) As x approaches c, the values of $|R(x)| \to \infty$ $[\lim_{x \to c^-} R(x) = \infty;$ $\lim_{x \to c^+} R(x) = \infty]$. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



(d) As x approaches c, the values of $|R(x)| \to \infty$ $[\lim_{x \to c^-} R(x) = -\infty;$ $\lim_{x \to c^+} R(x) = \infty]$. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Oblique asymptote



OBJECTIVE 2

2 Find the Vertical Asymptotes of a Rational Function

Theorem

Locating Vertical Asymptotes

A rational function $R(x) = \frac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \frac{p(x)}{q(x)}$, in lowest terms, then R will have the vertical asymptote $x = r$.

EXAMPLE Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

$$R(x) = \frac{5x^2}{3+x}$$

$$H(x) = \frac{x-3}{(x+2)(x-2)}$$

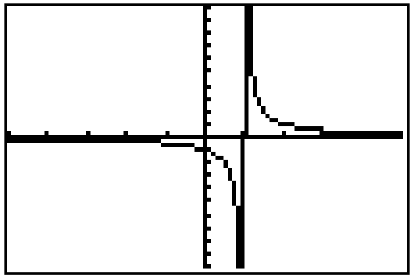
$$F(x) = \frac{x-1}{x^2+5x+4}$$

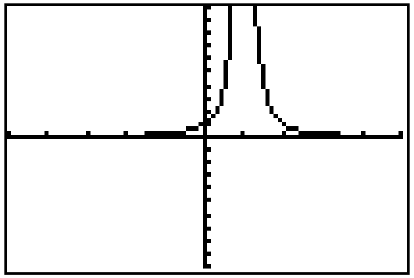
$$G(x) = \frac{x^2+3x+2}{x^2-4}$$

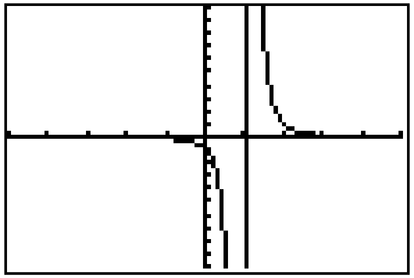
Exploration

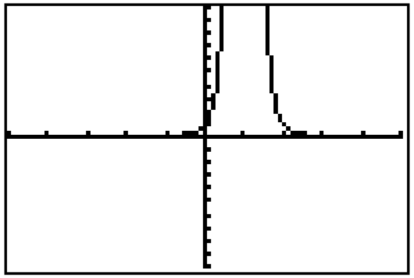
Graph each of the following rational functions:

$$R(x) = \frac{1}{x-1} \quad R(x) = \frac{1}{(x-1)^2} \quad R(x) = \frac{1}{(x-1)^3} \quad R(x) = \frac{1}{(x-1)^4}$$









Each has the vertical asymptote $x = 1$. What happens to the value of $R(x)$ as x approaches 1 from the right side of the vertical asymptote; that is, what is $\lim_{x \to 1^+} R(x)$? What happens to the value of $R(x)$ as x approaches 1 from the left side of the vertical asymptote; that is, what is $\lim_{x \to 1^-} R(x)$? How does the multiplicity of the zero in the denominator affect the graph of R?

OBJECTIVE 3

3 **Find the Horizontal or Oblique Asymptotes of a Rational Function**

Theorem

If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

EXAMPLE Finding Horizontal Asymptotes

Find the horizontal asymptotes, if any, of the graph of

$$R(x) = \frac{x+3}{x^2+2x+5}$$

If a rational function $R(x) = \dfrac{p(x)}{q(x)}$ is **improper**,

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

1. If $f(x) = b$, a constant, then the line $y = b$ is a horizontal asymptote of the graph of R.
2. If $f(x) = ax + b$, $a \neq 0$, then the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$H(x) = \frac{2x^2 + 7x - 1}{x^2 + 2}$$

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x)=\frac{5x^6-4x^3+3}{2x^6-5x^5+8x^4-7x^3+2}$$

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$G(x) = \frac{x^2 + 4x + 1}{x - 2}$$

SUMMARY Finding Horizontal and Oblique Asymptotes of a Rational Function R

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).

2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.

(a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.

(b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.

(c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of R will behave like the graph of the quotient.

NOTE The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■