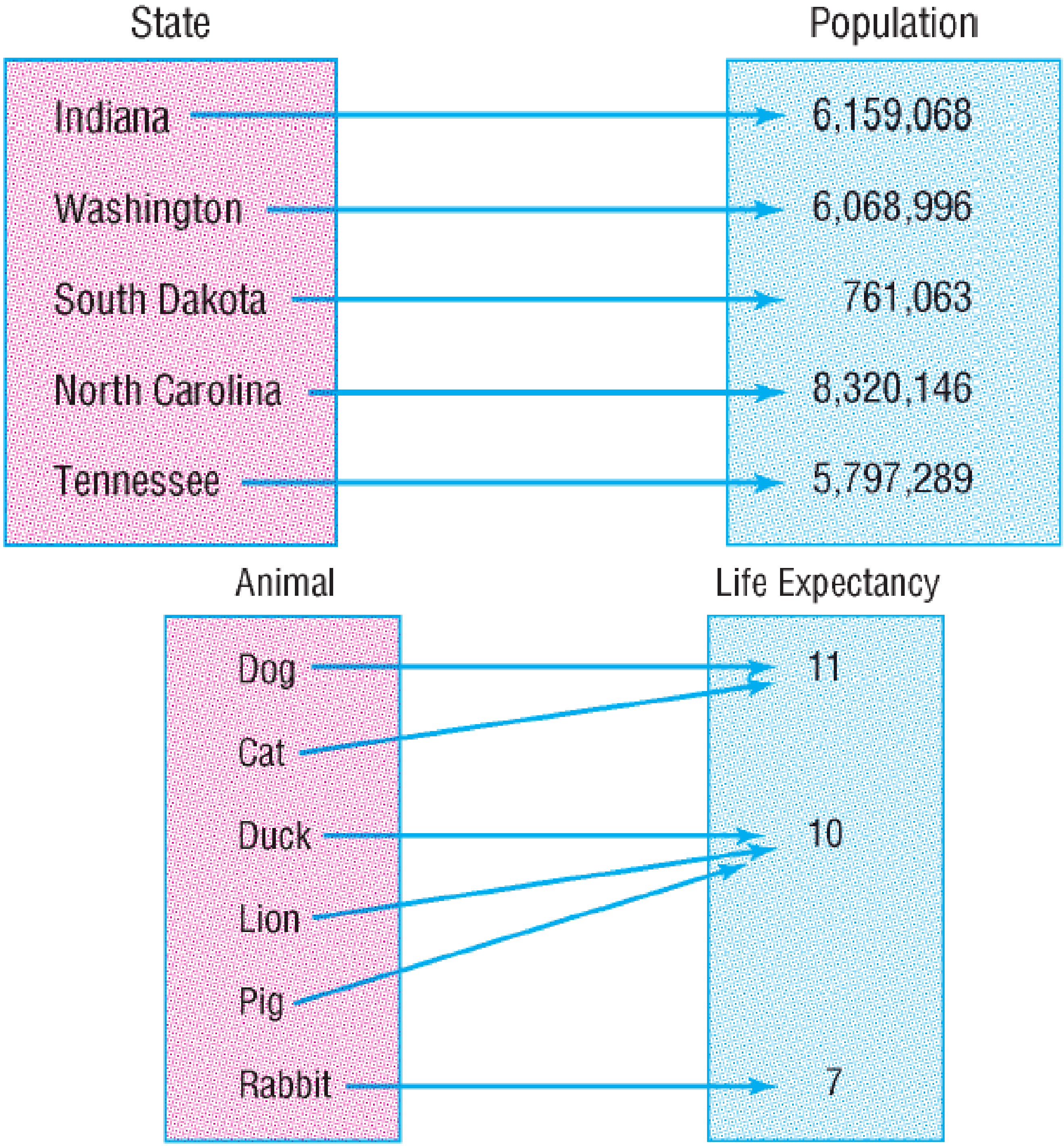
Section 6.2

One-to-One Functions;

Inverse Functions

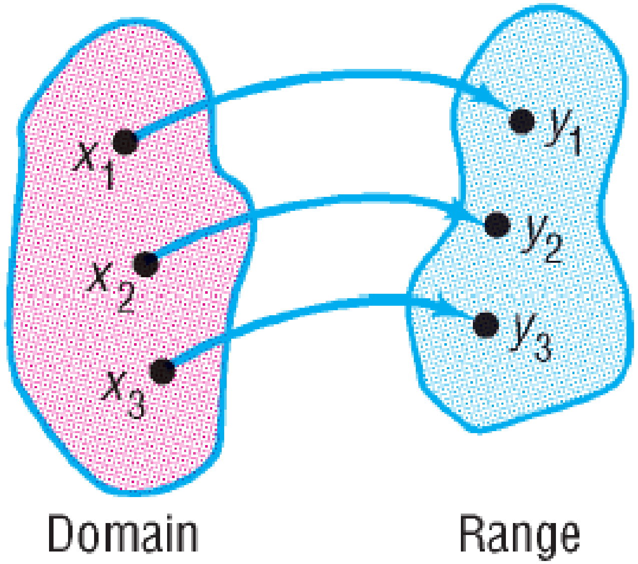
OBJECTIVE 1

1 Determine Whether a Function Is One-to-One

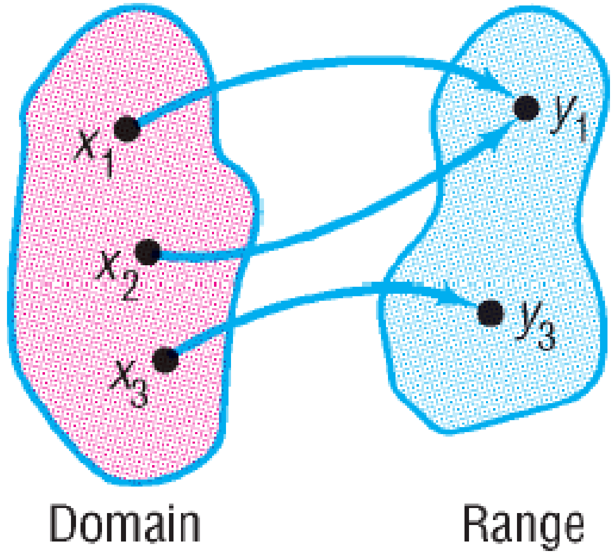

State
Population
Indiana
Washington
South Dakota
North Carolina
Tennessee
6,159,068
6,068,996
761,063
8,320,146
5,797,289
Animal
Life Expectancy
Dog
Cat
Duck
Lion
Pig
Rabbit
11
10
7

DEFINITION

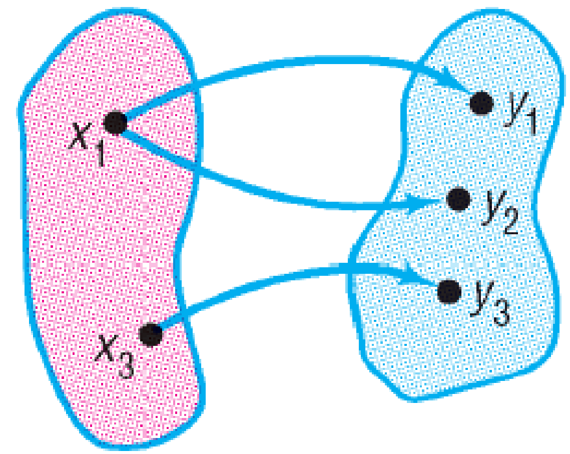
A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.



One-to-one function:
Each x in the domain has one and only one image in the range. No y in the range is the image of more than one x.



Not a one-to-one function:
y_1 is the image of both x_1 and x_2



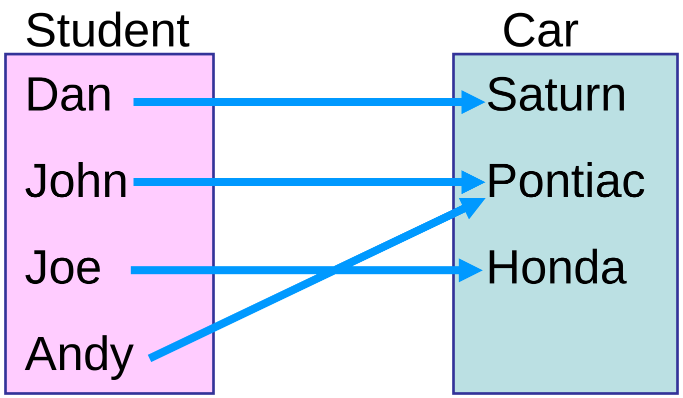
Not a function:
x_1 has two images, y_1 and y_2

EXAMPLE

Determining Whether a Function Is One-to-One

Determine whether the following functions are one-to-one.

(a)

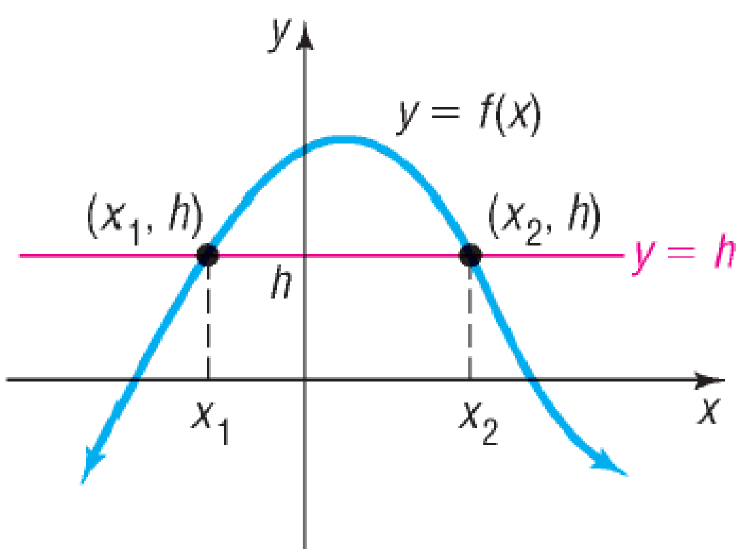


(b) $\{(1,5), (2,8), (3,11), (4,14)\}$

Theorem

Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

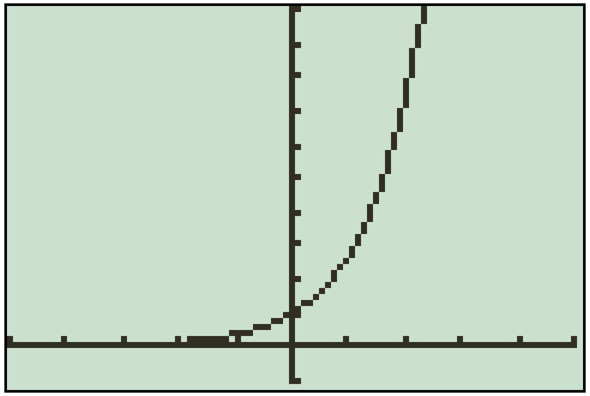


$f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$; f is not a one-to-one function.

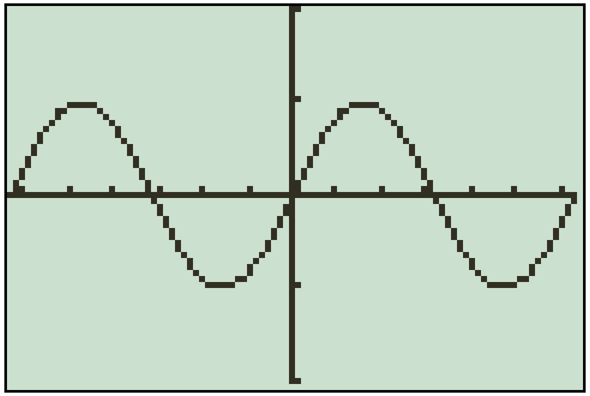
EXAMPLE Using the Horizontal-line Test

For each function, use the graph to determine whether the function is one-to-one.

(a)



(b)



Theorem

A function that is increasing on an interval I is a one-to-one function in I.

A function that is decreasing on an interval I is a one-to-one function on I.

OBJECTIVE 2

2 Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

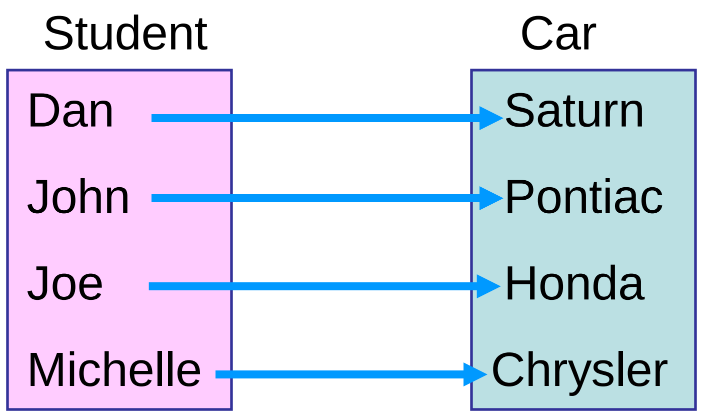
DEFINITION

Suppose f is a one-to-one function. Then, to each x in the domain of f, there is exactly one y in the range (because f is a function); and to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of f.** We use the symbol f^{-1} to denote the inverse of f.

EXAMPLE

Finding the Inverse of a Function Defined by a Map

Find the inverse of the following function. Let the domain of the function represent certain students, and let the range represent the make of that student's car. State the domain and the range of the inverse function.

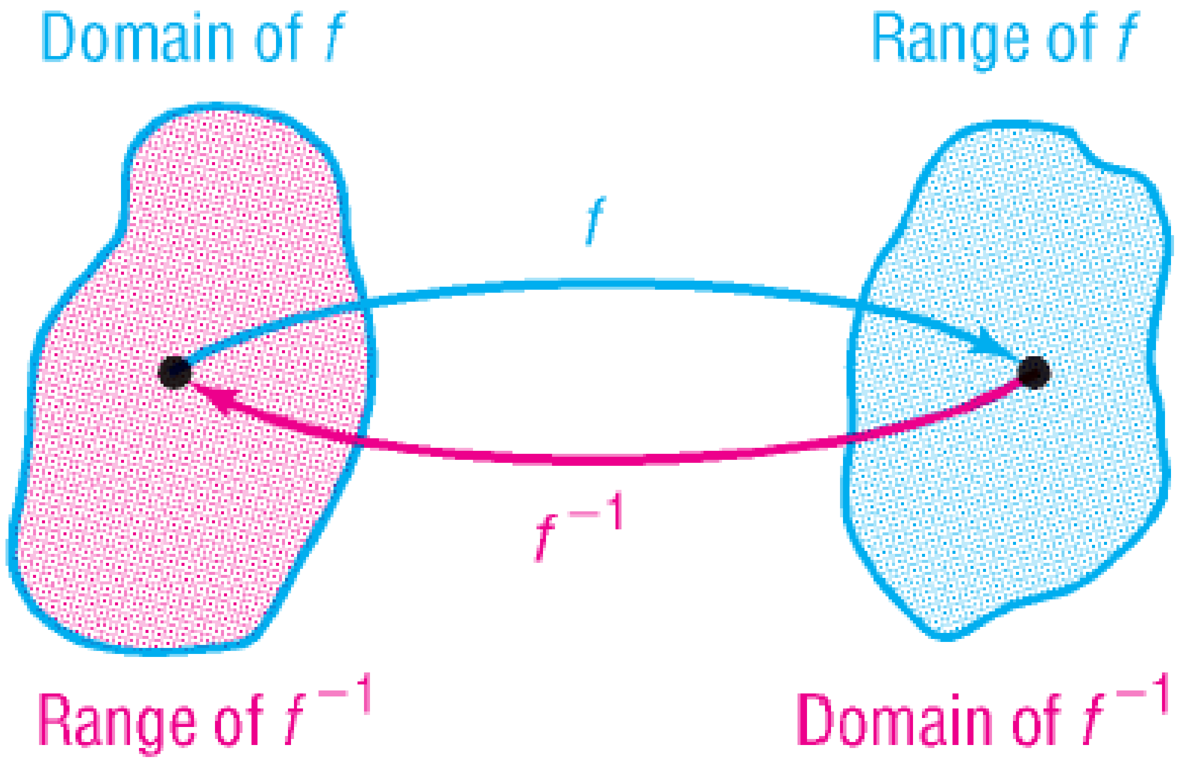


EXAMPLE Finding the Inverse of a Function Defined by a Set of Ordered Pairs

Find the inverse of the following one-to-one function:

$$\{(1,5), (2,8), (3,11), (4,14)\}$$

State the domain and the range of the function and its inverse.



Domain of f = Range of f^{-1} Range of f = Domain of f^{-1}

$$\boxed{\text{Input } x} \xrightarrow{\text{Apply } f} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(f(x)) = x}$$

$$\boxed{\text{Input } x} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f} \boxed{f(f^{-1}(x)) = x}$$

$$f^{-1}(f(x)) = x \quad \text{where } x \text{ is in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{where } x \text{ is in the domain of } f^{-1}$$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & f(x) = 2x \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$$f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$

EXAMPLE Verifying Inverse Functions

Verify that the inverse of $g(x) = x^3 + 2$ is $g^{-1}(x) = \sqrt[3]{x-2}$ by showing that $g\left(g^{-1}(x)\right) = x$ for all x in the domain of g and $g^{-1}\left(g(x)\right) = x$ for all x in the domain of g^{-1}.

EXAMPLE **Verifying Inverse Functions**

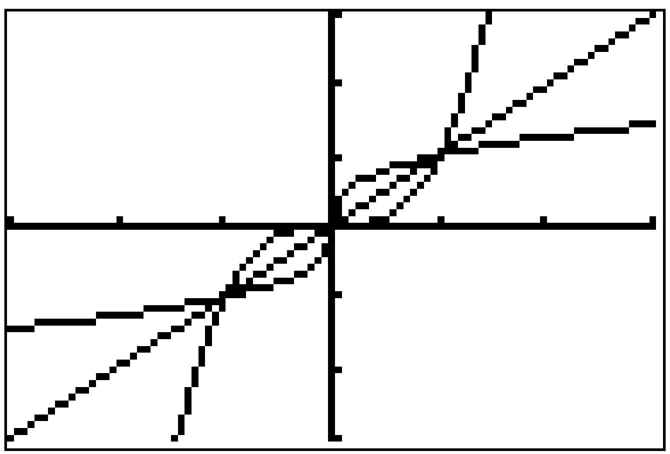
Verify that the inverse of $f\left(x\right)=\dfrac{1}{2x+1}$ is $f^{-1}\left(x\right)=\dfrac{1}{2x}-\dfrac{1}{2}$.

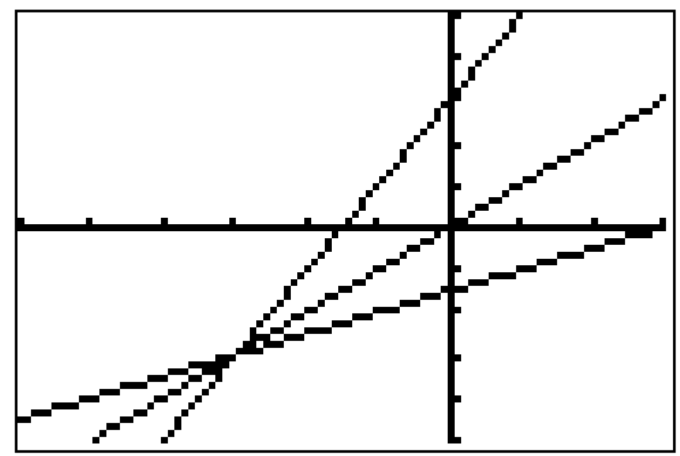
For what values of x is $f\left(f^{-1}\left(x\right)\right)=x$?

For what values of x is $f^{-1}\left(f\left(x\right)\right)=x$?

Exploration

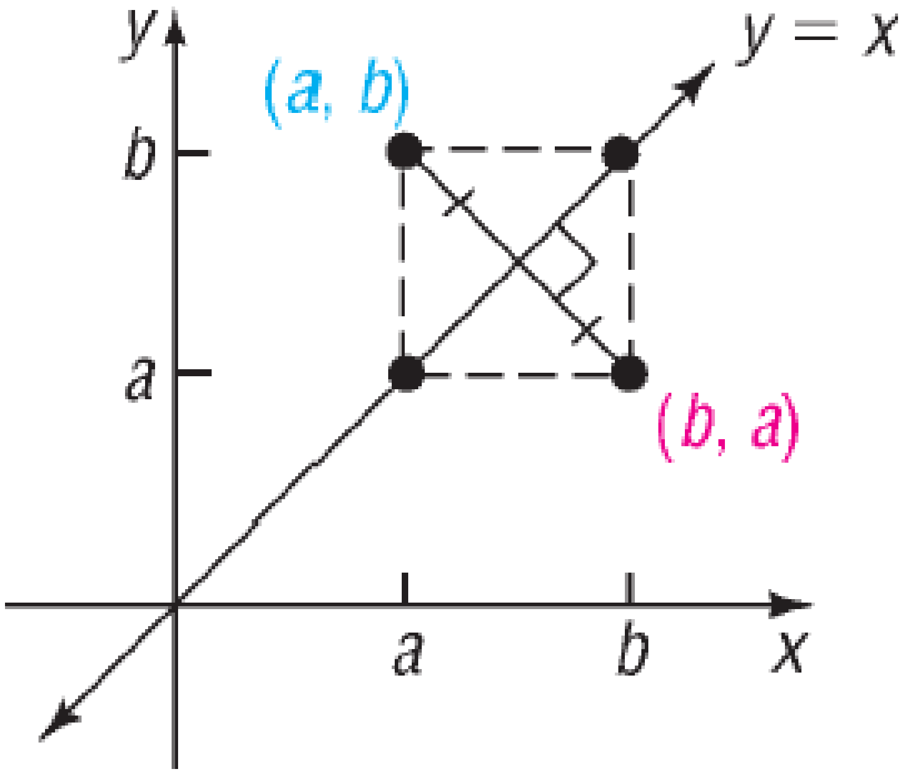
Simultaneously graph $Y_1 = x$, $Y_2 = x^3$, and $Y_3 = \sqrt[3]{x}$ on a square screen with $-3 \le x \le 3$. What do you observe about the graphs of $Y_2 = x^3$, its inverse $Y_3 = \sqrt[3]{x}$, and the line $Y_1 = x$? Repeat this experiment by simultaneously graphing $Y_1 = x$, $Y_2 = 2x + 3$, and $Y_3 = \frac{1}{2}(x - 3)$ on a square screen with $-6 \le x \le 3$. Do you see the symmetry of the graph of Y_2 and its inverse Y_3 with respect to the line $Y_1 = x$?





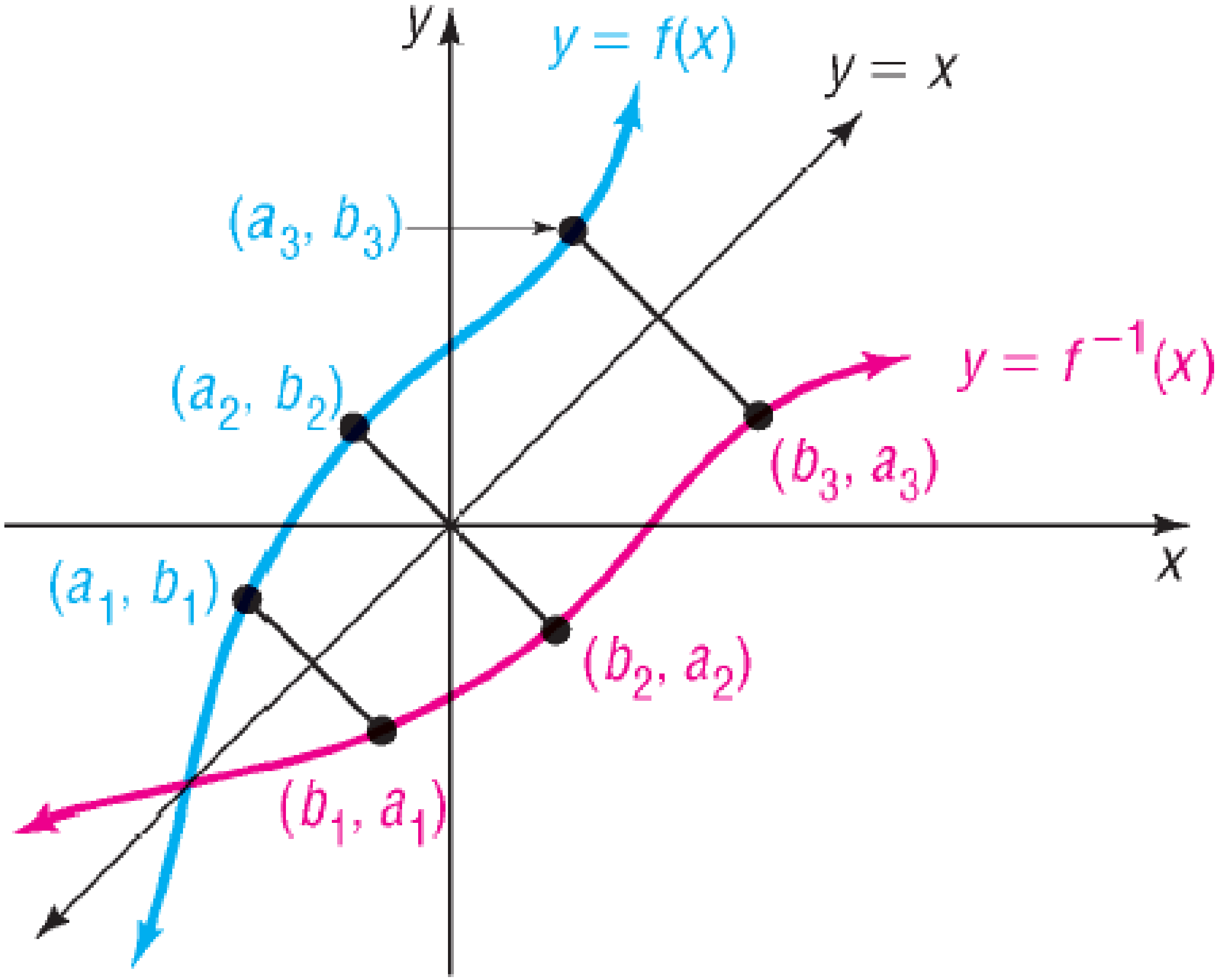
OBJECTIVE 3

3 Obtain the Graph of the Inverse Function from the Graph of the Function



Theorem

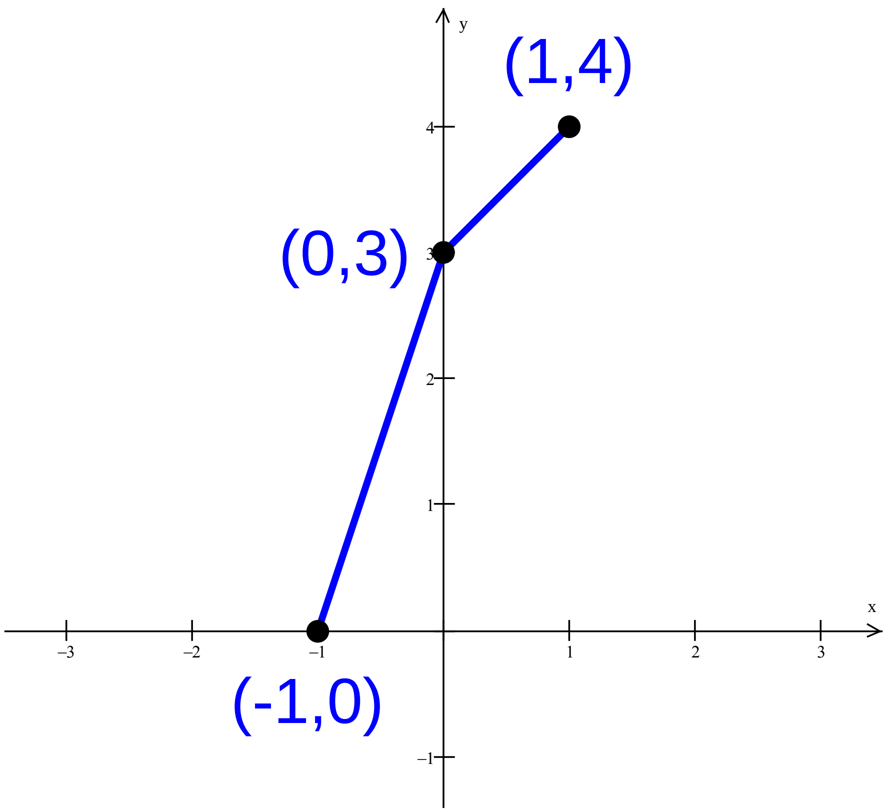
The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.



y
y = f(x)
y = x
(a_3, b_3)
(a_2, b_2)
$y = f^{-1}(x)$
(b_3, a_3)
x
(a_1, b_1)
(b_2, a_2)
(b_1, a_1)

EXAMPLE Graphing the Inverse Function

The graph shown is that of a one-to-one function. Draw the graph of the inverse.



OBJECTIVE 4

4 Find the Inverse of a Function Defined by an Equation

EXAMPLE

Finding the Inverse Function

Find the inverse of $f(x) = -\frac{1}{3}x + 1.$

Graph f and f^{-1} on the same coordinate axes.

Step-by-Step Solution

STEP 1 Replace $f(x)$ with y. In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.

STEP 2 If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}, $y = f^{-1}(x)$.

STEP 3 Check the result by showing that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Procedure for Finding the Inverse of a One-to-One Function

STEP 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

STEP 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}

$$y = f^{-1}(x)$$

STEP 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE Finding the Inverse Function

The function

$$f(x) = \frac{2x-1}{x+1}, \qquad x \neq -1$$

is one-to-one. Find its inverse and check the result.

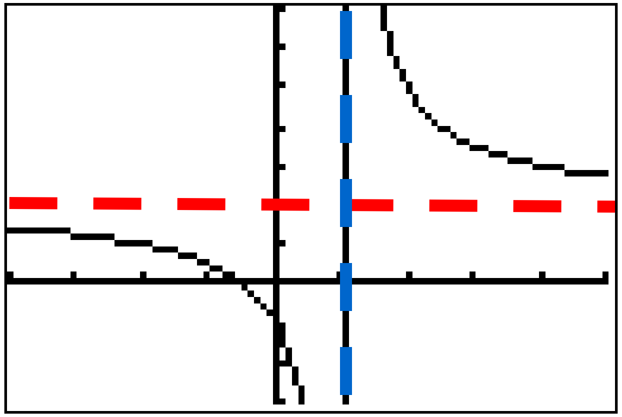
Exploration

$$\text{if } f(x) = \frac{2x+1}{x-1}, \text{ then } f^{-1}(x) = \frac{x+1}{x-2}.$$

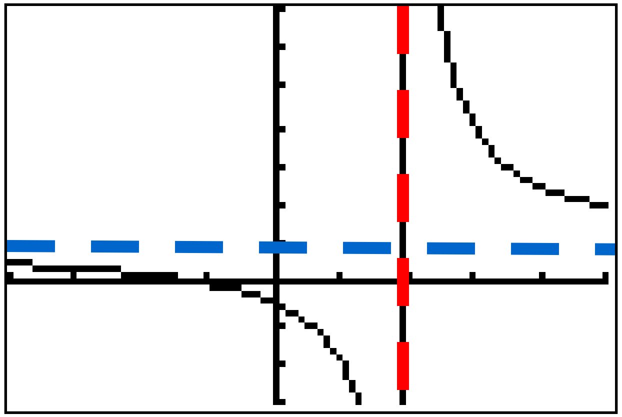
Compare the vertical and horizontal asymptotes of f and f^{-1}.

. What did you find? Are you surprised?

$$f(x) = \frac{2x+1}{x-1}$$



$$f^{-1}(x) = \frac{x+1}{x-2}$$



EXAMPLE Finding the Range of a Function

Find the domain and the range of

$$f(x) = \frac{2x-1}{x+1}$$

EXAMPLE

Finding the Inverse of a Domain-restricted Function

Find the inverse of $y=(x-1)^2$ if $x \geq 1$.

Summary

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$

   for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.
5. To find the range of a one-to-one function f, find the domain of the inverse function f^{-1}.