Section 6.3
Exponential Functions
OBJECTIVE 1

1 Evaluate Exponential Functions
Using a Calculator to Evaluate Powers of 2

Using a calculator, evaluate:

(a) \(2^{1.4}\)   (b) \(2^{1.41}\)   (c) \(2^{1.414}\)   (d) \(2^{1.4142}\)   (e) \(2^{\sqrt{2}}\)
Theorem

Laws of Exponents

If $s$, $t$, $a$, and $b$ are real numbers with $a > 0$ and $b > 0$, then

\[
\begin{align*}
    a^s \cdot a^t &= a^{s+t} \\
    (a^s)^t &= a^{st} \\
    (ab)^s &= a^s \cdot b^s \\
    1^s &= 1 \\
    a^{-s} &= \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \\
    a^0 &= 1
\end{align*}
\]
Introduction to Exponential Growth

Let’s examine a function $f$ that has the following two properties:

1. The value of $f$ doubles with every 1-unit increase in the independent variable $x$.
2. The value of $f$ at $x = 0$ is 5, so $f(0) = 5$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
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**DEFINITION**

An *exponential function* is a function of the form

\[ f(x) = Ca^x \]

where \(a\) is a positive real number \((a > 0)\) and \(a \neq 1\), and \(C \neq 0\) is a real number. The domain of \(f\) is the set of all real numbers. The base \(a\) is the **growth factor**, and because \(f(0) = Ca^0 = C\), we call \(C\) the **initial value**.
Theorem

For an exponential function \( f(x) = C \cdot a^x, a > 0, a \neq 1 \), if \( x \) is any real number, then

\[
\frac{f(x + 1)}{f(x)} = a \quad \text{or} \quad f(x + 1) = af(x)
\]
EXAMPLE Identifying Linear or Exponential Functions

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

<table>
<thead>
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<tr>
<td>2</td>
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(b)

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(c)

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OBJECTIVE 2

2  Graph Exponential Functions
Graph the exponential function: \( f(x) = 2^x \)
Properties of the Exponential Function $f(x) = a^x, a > 1$

1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no $x$-intercepts; the $y$-intercept is 1.
3. The $x$-axis ($y = 0$) is a horizontal asymptote as $x \to -\infty$.
4. $f(x) = a^x, a > 1$, is an increasing function and is one-to-one.
5. The graph of $f$ contains the points $(0, 1), (1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of $f$ is smooth and continuous, with no corners or gaps.
Graph the exponential function: \( f(x) = \left( \frac{1}{2} \right)^x \)
Seeing the Concept

Using a graphing utility, simultaneously graph:

(a) \( Y_1 = 3^x, \ Y_2 = \left( \frac{1}{3} \right)^x \)

(b) \( Y_1 = 6^x, \ Y_2 = \left( \frac{1}{6} \right)^x \)
The graph shows two exponential functions:

1. $y_1 = \left(\frac{1}{3}\right)^x$
2. $y_2 = \left(\frac{1}{6}\right)^x$

The points on the graph include:

- $(-1, 6)$
- $(0, 1)$
- $(1, \frac{1}{3})$
- $(1, \frac{1}{6})$

The graph also includes a horizontal line at $y = 0$. The graph is split into three sections, each showing the behavior of $y_1$ and $y_2$ for different ranges of $x$.
Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no $x$-intercepts; the $y$-intercept is 1.
3. The $x$-axis ($y = 0$) is a horizontal asymptote as $x \to \infty$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of $f$ contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of $f$ is smooth and continuous, with no corners or gaps.
Graph $f(x) = 2 \cdot 3^{x+1} - 4$ and determine the domain, range, and horizontal asymptote of $f$. 
OBJECTIVE 3

3  Define the Number e
The **number** $e$ is defined as the number that the expression 

\[
\left(1 + \frac{1}{n}\right)^n
\]

approaches as $n \to \infty$.

In calculus, this is expressed using limit notation as

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{1}{n}$</th>
<th>$1 + \frac{1}{n}$</th>
<th>$\left(1 + \frac{1}{n}\right)^n$</th>
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<tbody>
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<td>$1 + 10^{-9}$</td>
<td>2.718281827</td>
</tr>
</tbody>
</table>
\[ y = e^x \]

- Table:
  - \( x = -2 \) : \( y = 0.13534 \)
  - \( x = -1 \) : \( y = 0.36788 \)
  - \( x = 0 \) : \( y = 1 \)
  - \( x = 1 \) : \( y = 2.7183 \)
  - \( x = 2 \) : \( y = 7.3891 \)

- Graphs:
  - \( Y_1 = e^x \)
  - \( Y_2 = e^x \)
  - \( Y_3 = 3^x \)
  - \( Y_4 = 2^x \)
Graphing Exponential Functions Using Transformations

Graph \( f(x) = -e^{x-2} \) and determine the domain, range, and horizontal asymptote of \( f \).
OBJECTIVE 4

4  Solve Exponential Equations
If \( a^u = a^v \), then \( u = v \)
EXAMPLE

Solving an Exponential Equation

Solve: \( 2^{3x-1} = 32 \)

If \( a^u = a^v \), then \( u = v \)
EXAMPLE  Solving an Exponential Equation

Solve: \( e^{2x-1} = \frac{1}{e^{3x}} \cdot \left( e^{-x} \right)^4 \)

If \( a^u = a^v \), then \( u = v \)
EXAMPLE  Exponential Probability

Between 9:00 PM and 10:00 PM cars arrive at Burger King’s drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within $t$ minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

(c) Graph $F$ using your graphing utility.

(d) What value does $F$ approach as $t$ becomes unbounded in the positive direction?
Summary

Properties of the Exponential Function

\[ f(x) = a^x, \quad a > 1 \]

Domain: the interval \((-\infty, \infty)\); Range: the interval \((0, \infty)\)
\(x\)-intercepts: none; \(y\)-intercept: 1
Horizontal asymptote: \(x\)-axis \((y = 0)\) as \(x \to -\infty\)
Increasing: one-to-one; smooth; continuous

\[ f(x) = a^x, \quad 0 < a < 1 \]

Domain: the interval \((-\infty, \infty)\); Range: the interval \((0, \infty)\).
\(x\)-intercepts: none; \(y\)-intercept: 1
Horizontal asymptote: \(x\)-axis \((y = 0)\) as \(x \to \infty\)
Decreasing: one-to-one; smooth; continuous

If \(a^u = a^v\), then \(u = v\).