Section 6.3

Exponential Functions

OBJECTIVE 1

1 Evaluate Exponential Functions

EXAMPLE

Using a Calculator to Evaluate Powers of 2

Using a calculator, evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Theorem

Laws of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s \cdot a^t = a^{s+t} \qquad (a^s)^t = a^{st} \qquad (ab)^s = a^s \cdot b^s$$

$$1^s = 1 \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1$$

Introduction to Exponential Growth

Let's examine a function f that has the following two properties:

1. The value of f doubles with every 1-unit increase in the independent variable x.
2. The value of f at $x = 0$ is 5, so $f(0) = 5$.

x	$f(x)$
0	5
1	10
2	20
3	40
4	80

DEFINITION

An **exponential function** is a function of the form

$$f(x) = Ca^x$$

where a is a positive real number ($a > 0$) and $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor,** and because $f(0) = Ca^0 = C$, we call C the **initial value.**

Theorem

For an exponential function $f(x) = C \cdot a^x$, $a > 0$, $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)$$

EXAMPLE Identifying Linear or Exponential Functions

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	y
−1	5
0	2
1	−1
2	−4
3	−7

(b)

x	y
−1	2
0	4
1	7
2	11
3	15

(c)

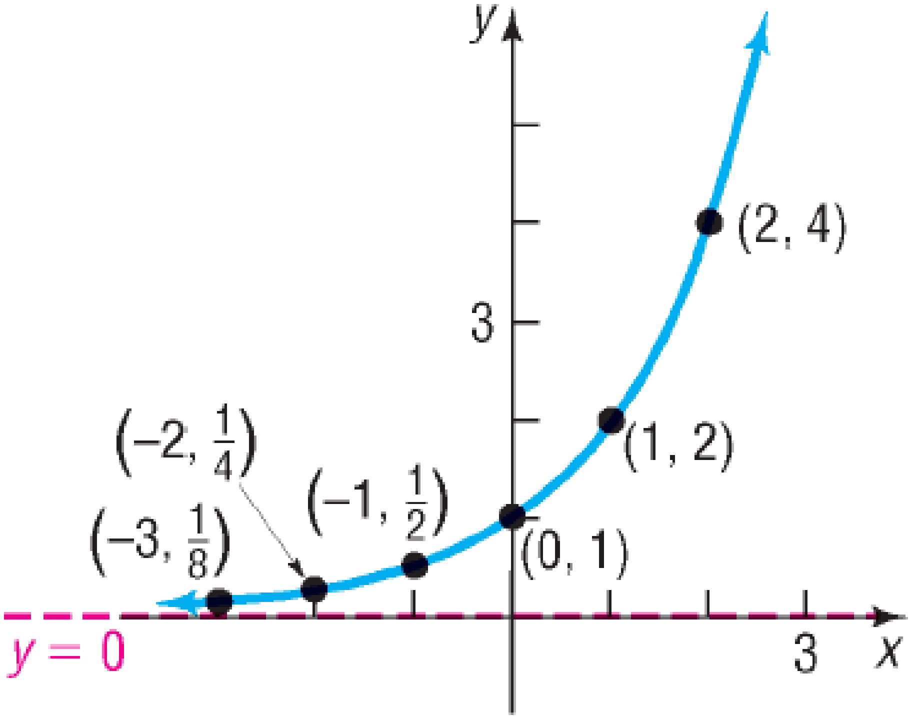
x	y
−1	32
0	16
1	8
2	4
3	2

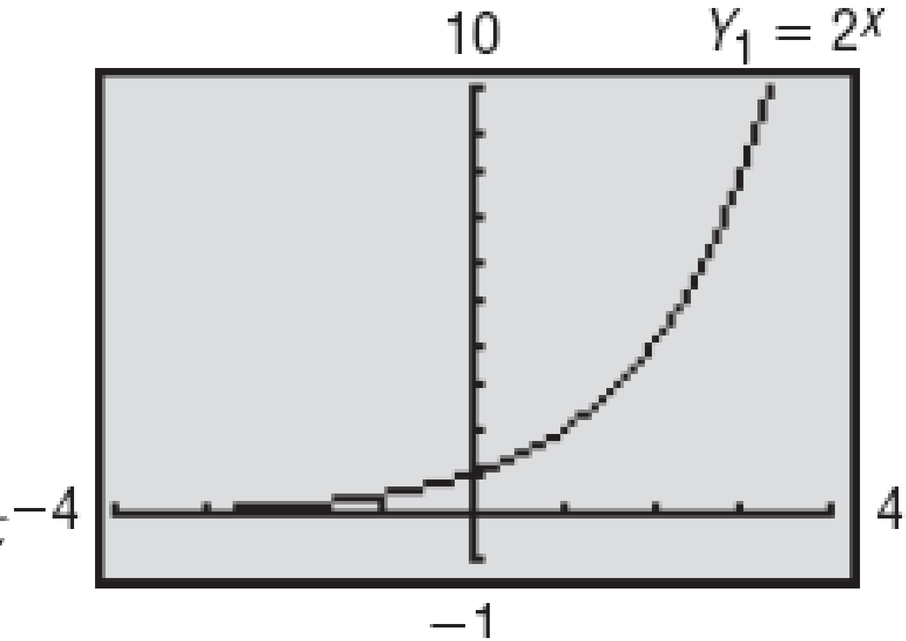
OBJECTIVE 2

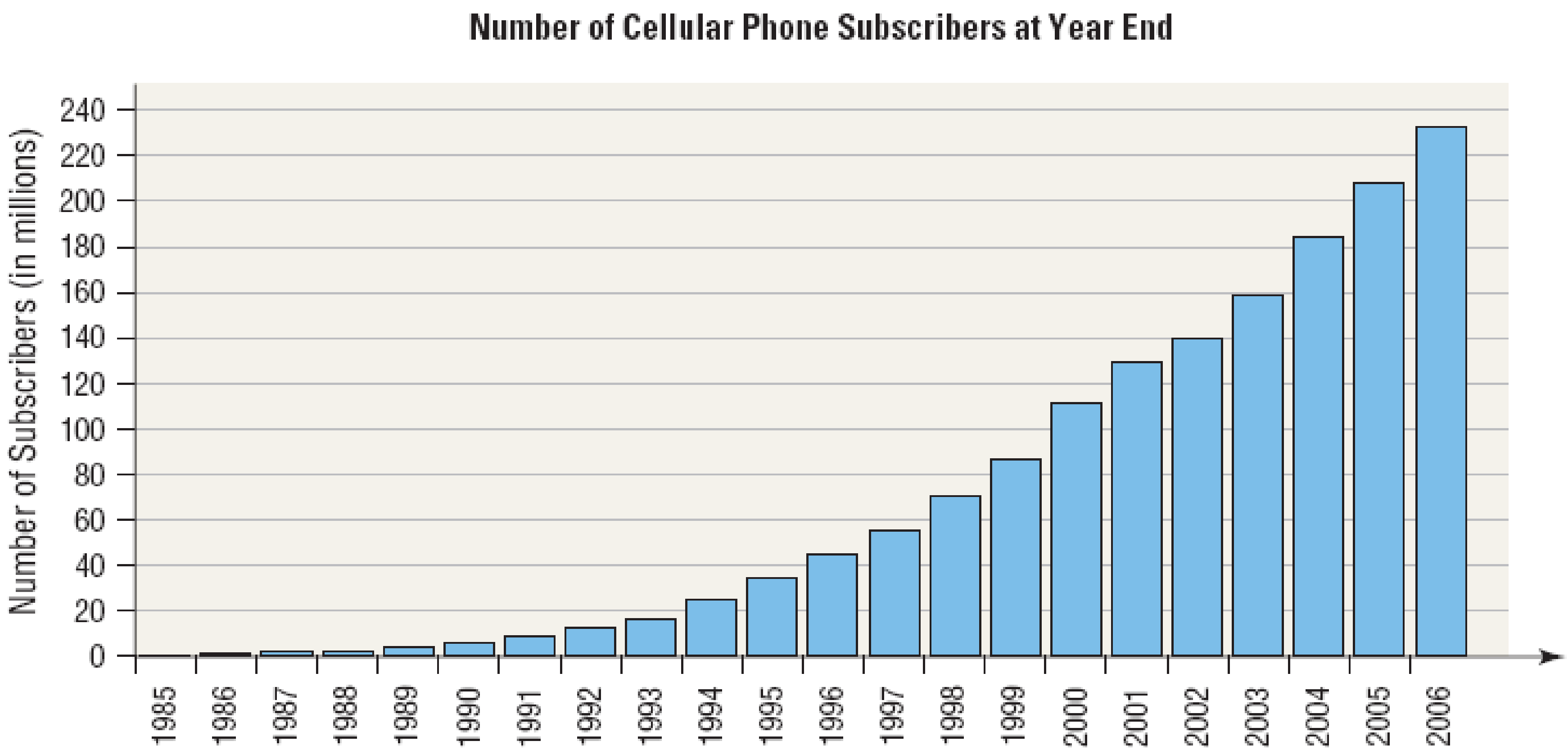
2 Graph Exponential Functions

EXAMPLE **Graphing an Exponential Function**

Graph the exponential function: $f(x) = 2^x$







Number of Cellular Phone Subscribers at Year End
Number of Subscribers (in millions)
240
220
200
180
160
140
120
100
80
60
40
20
0
1985
1986
1987
1988
1989
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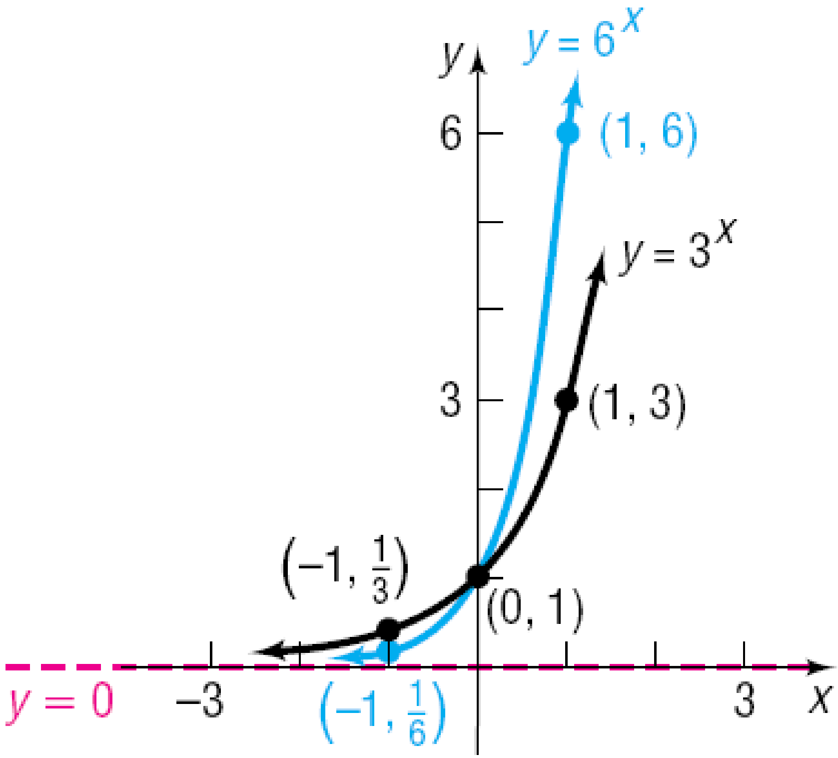
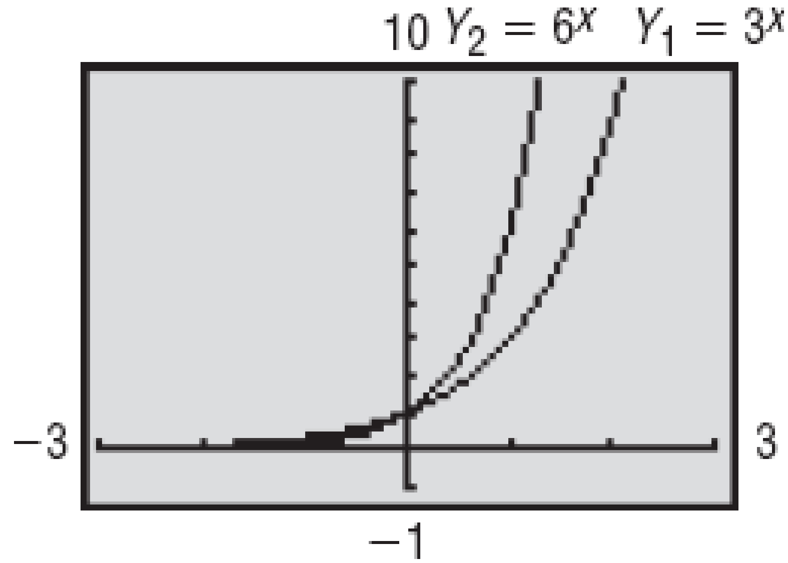
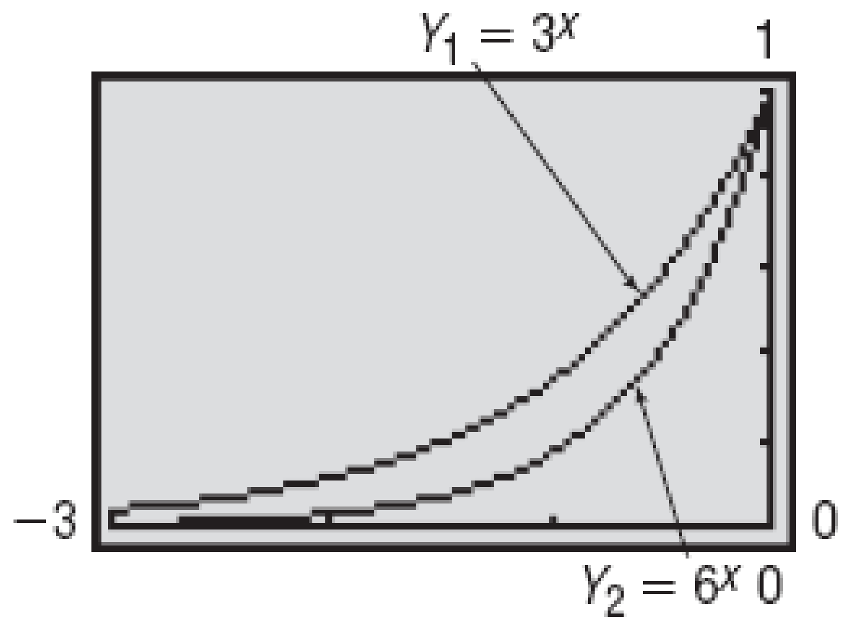
2002
2003
2004
2005
2006



y
y = 6^x
6
(1, 6)
y = 3^x
3
(1, 3)
(−1, 1/3)
(0, 1)
y = 0
−3
(−1, 1/6)
3
x



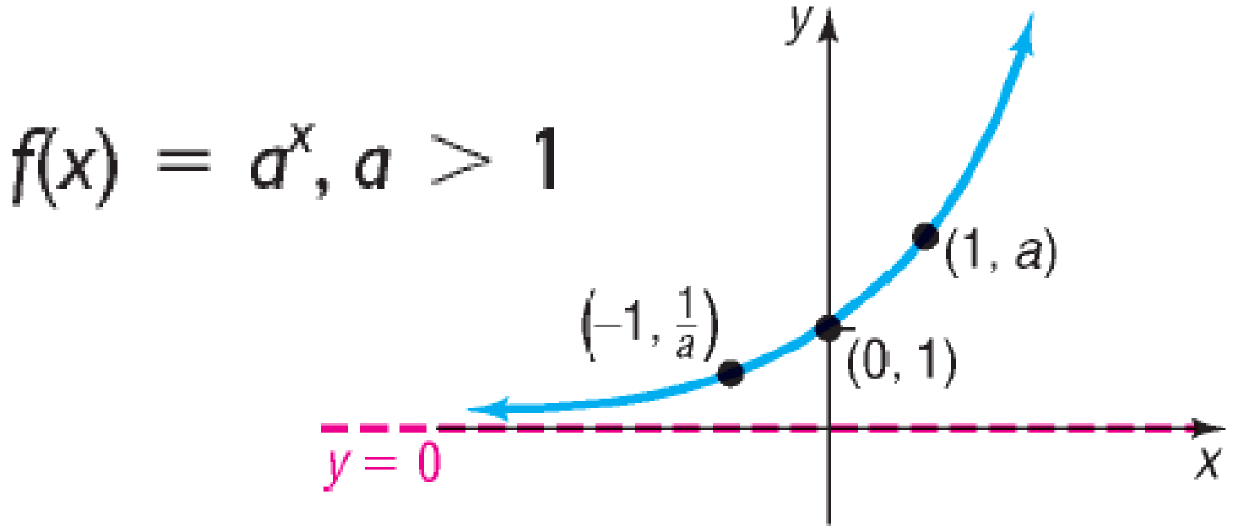
10 Y2 = 6^x
Y1 = 3^x
−3
3
−1



Y1 = 3^x
1
−3
0
Y2 = 6^x 0

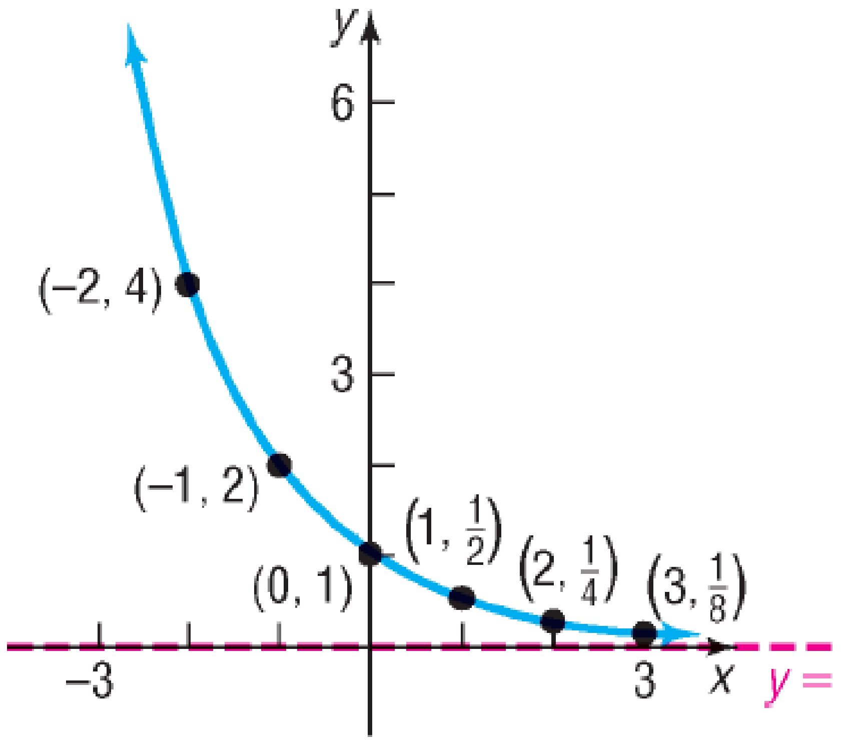
Properties of the Exponential Function $f(x) = a^x$, $a > 1$

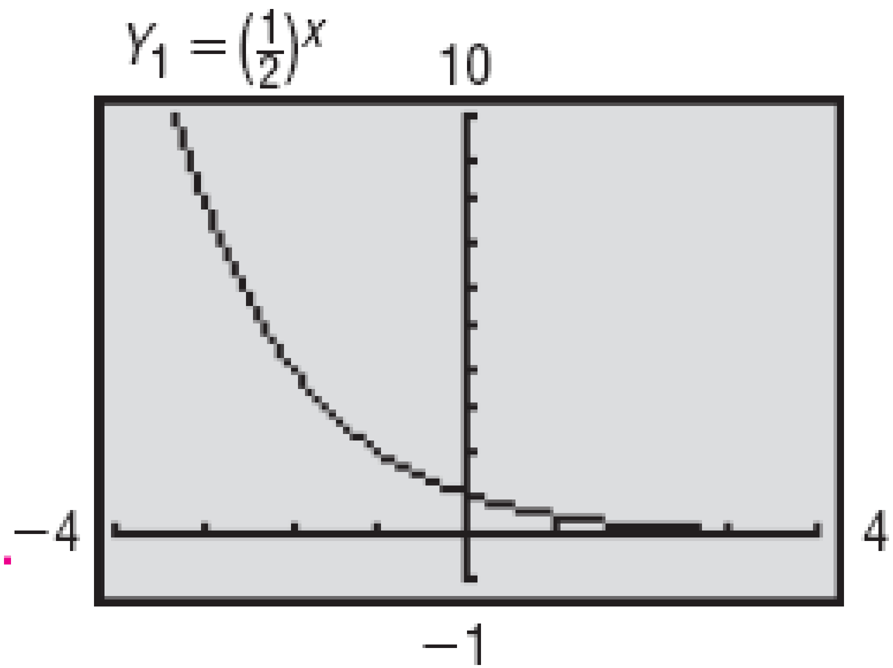
1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to -\infty$.
4. $f(x) = a^x$, $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps.



EXAMPLE **Graphing an Exponential Function**

Graph the exponential function: $f(x) = \left(\frac{1}{2}\right)^x$



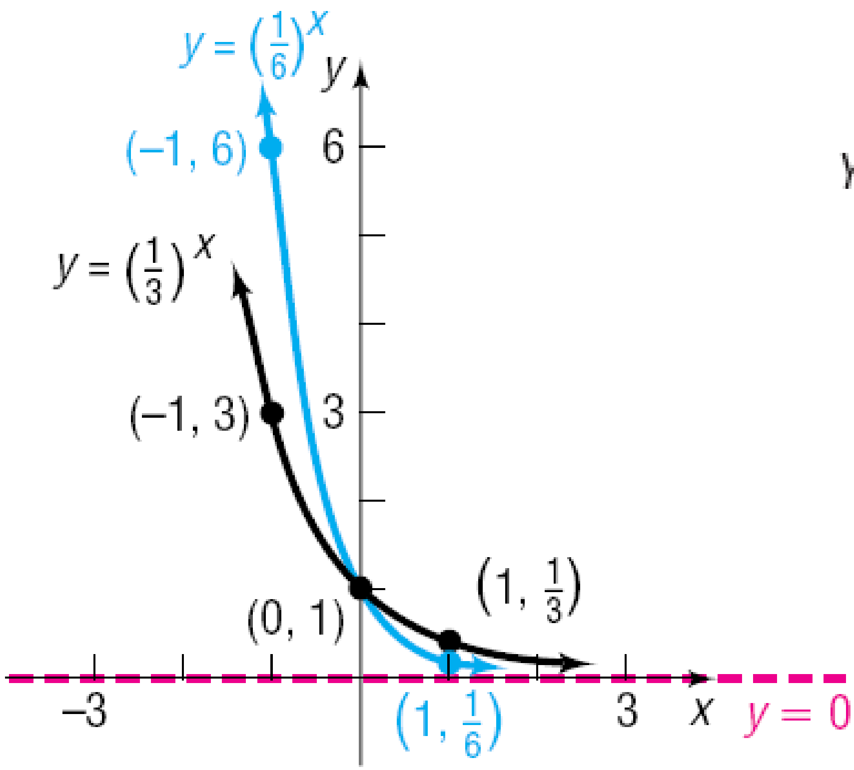


Seeing the Concept

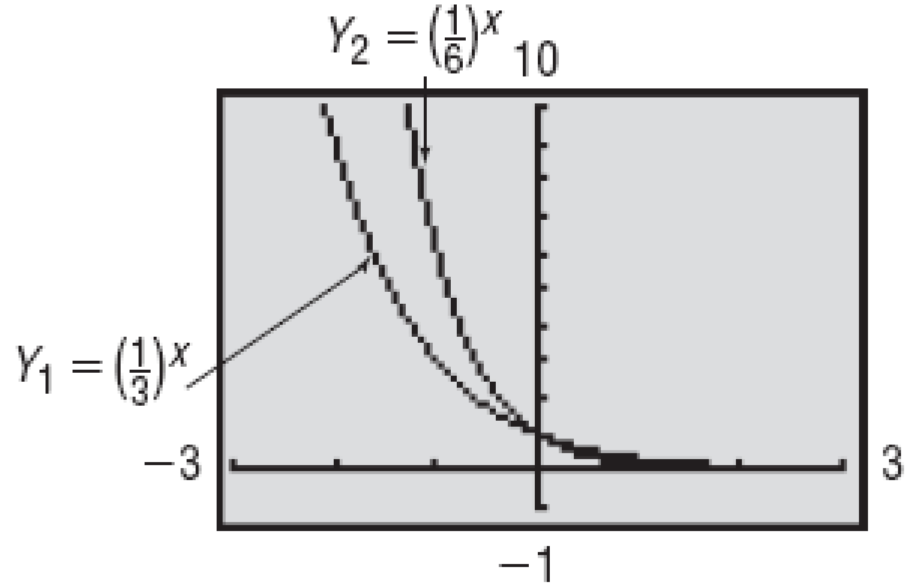
Using a graphing utility, simultaneously graph:

(a) $Y_1 = 3^x, Y_2 = \left(\frac{1}{3}\right)^x$

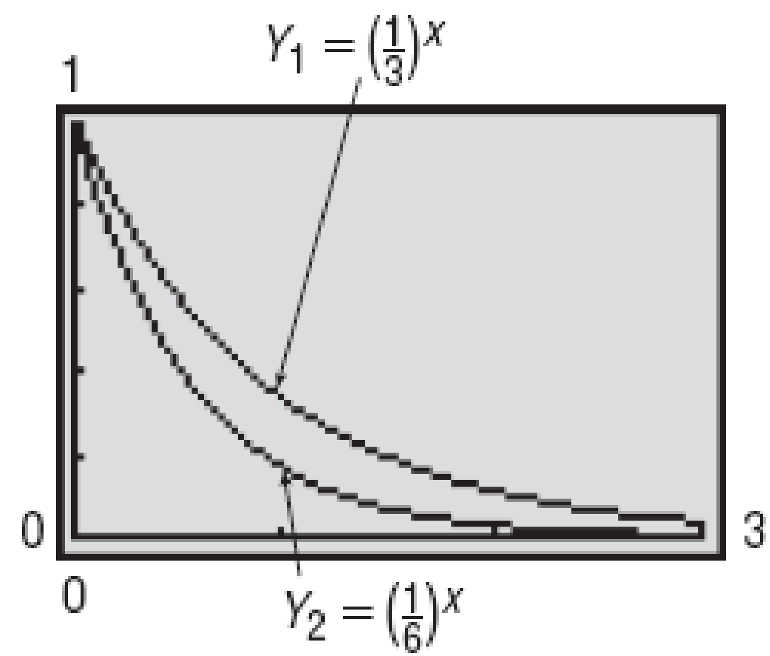
(b) $Y_1 = 6^x, Y_2 = \left(\frac{1}{6}\right)^x$



$y = \left(\frac{1}{6}\right)^x$
y
(–1, 6)
6
$y = \left(\frac{1}{3}\right)^x$
(–1, 3)
3
(0, 1)
$\left(1, \frac{1}{3}\right)$
–3
$\left(1, \frac{1}{6}\right)$
3
x
y = 0



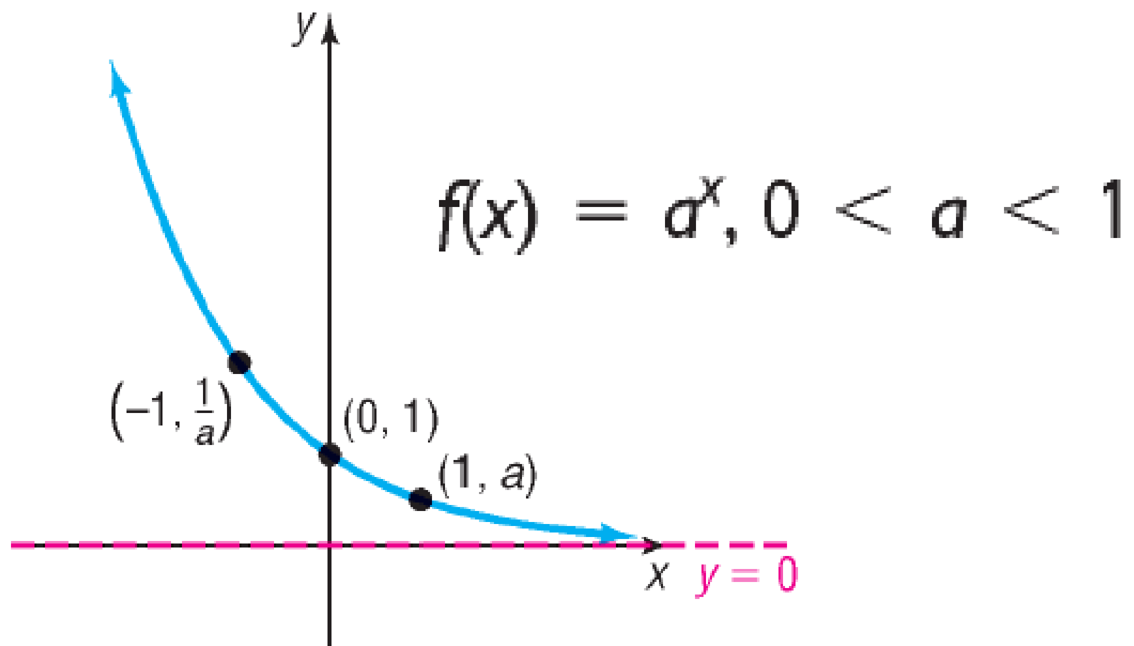
$Y_2 = \left(\frac{1}{6}\right)^x$
10
$Y_1 = \left(\frac{1}{3}\right)^x$
−3
3
−1



$Y_1 = \left(\frac{1}{3}\right)^x$
1
0
3
0
$Y_2 = \left(\frac{1}{6}\right)^x$

Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to \infty$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps.



EXAMPLE

Graphing Exponential Functions Using Transformations

Graph $f(x) = 2 \cdot 3^{x+1} - 4$ and determine the domain, range, and horizontal asymptote of f.

OBJECTIVE 3

3 Define the Number *e*

DEFINITION

The **number *e*** is defined as the number that the expression

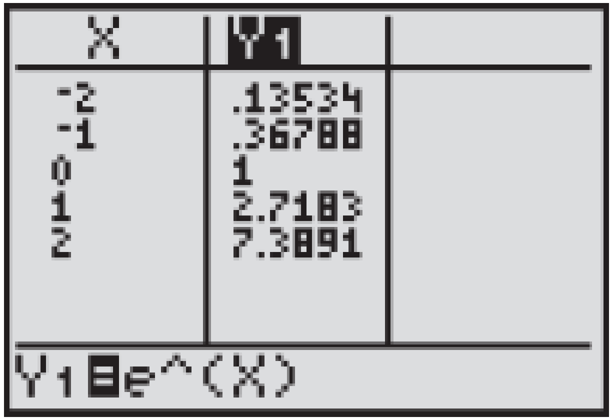
$$\left(1 + \frac{1}{n}\right)^n$$

approaches as $n \to \infty$.

In calculus, this is expressed using limit notation as

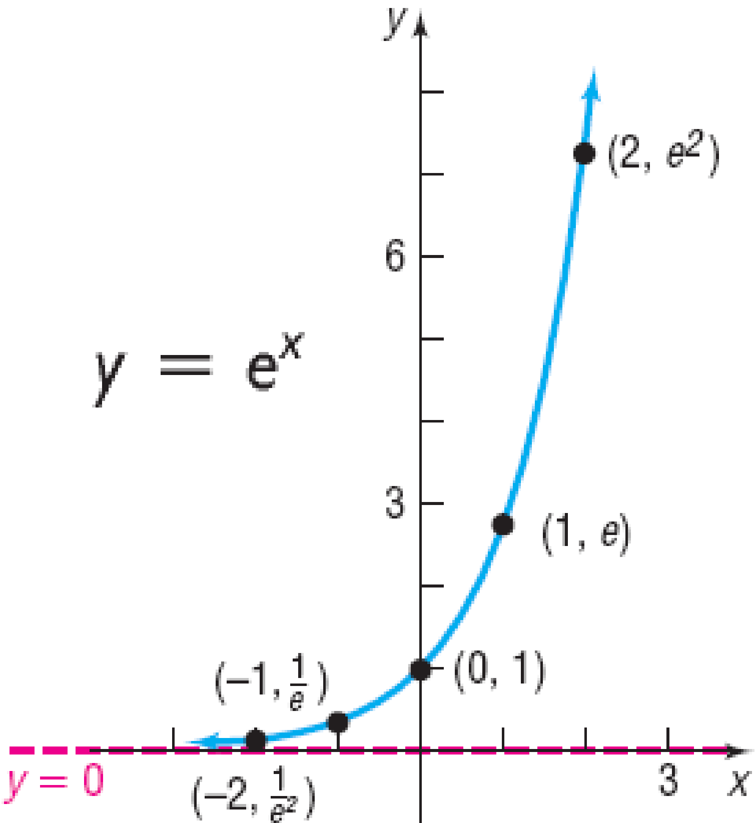
$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

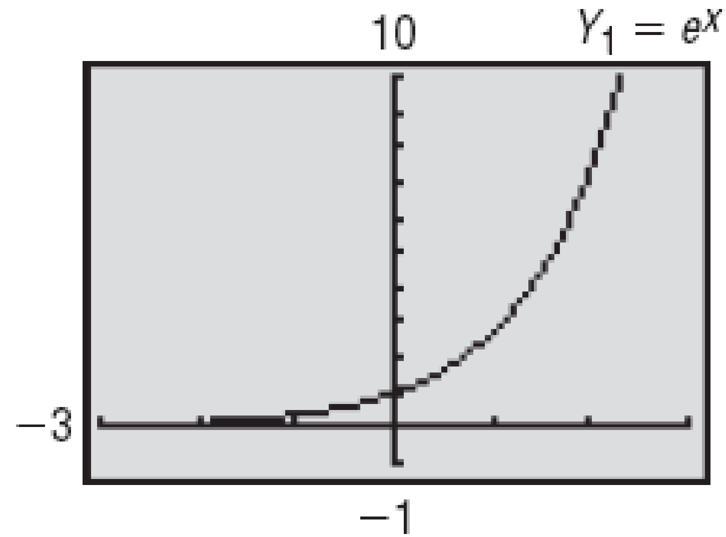
n	$\frac{1}{n}$	$1 + \frac{1}{n}$	$\left(1 + \frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
1,000,000,000	10^{-9}	$1 + 10^{-9}$	2.718281827

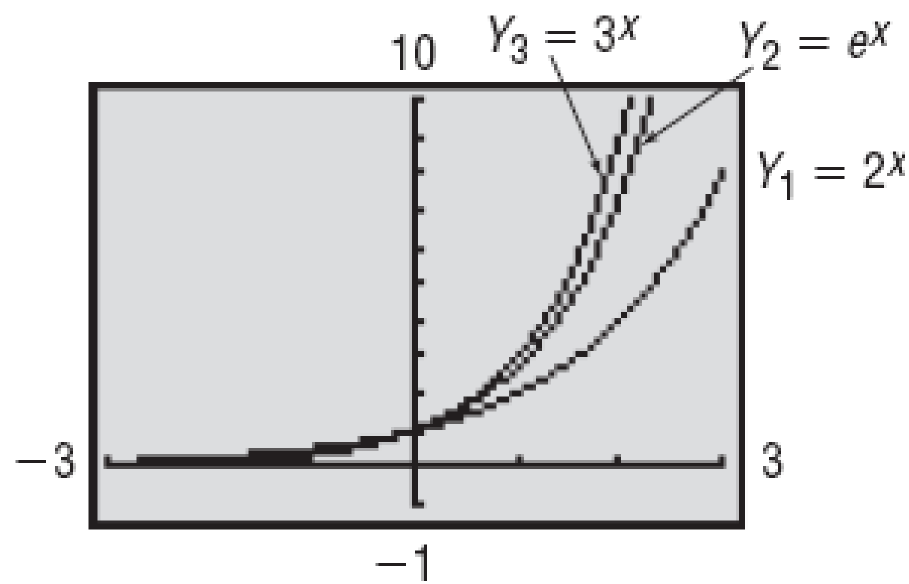


X	Y1
-2	.13534
-1	.36788
0	1
1	2.7183
2	7.3891

Y1=e^(X)







EXAMPLE

Graphing Exponential Functions Using Transformations

Graph $f(x) = -e^{x-2}$ and determine the domain, range, and horizontal asymptote of f.

OBJECTIVE 4

4 Solve Exponential Equations

$$\text{If} \quad a^u = a^v, \qquad \text{then} \quad u = v$$

EXAMPLE Solving an Exponential Equation

$$\text{Solve: } 2^{3x-1} = 32$$

If $a^u = a^v$, then $u = v$

EXAMPLE **Solving an Exponential Equation**

Solve: $e^{2x-1} = \frac{1}{e^{3x}} \cdot \left(e^{-x}\right)^4$

If $a^u = a^v$, then $u = v$

EXAMPLE Exponential Probability

Between 9:00 PM and 10:00 PM cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

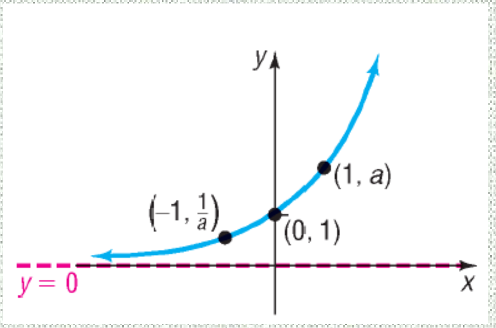
(c) Graph F using your graphing utility.

(d) What value does F approach as t becomes unbounded in the positive direction?

Summary

Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$



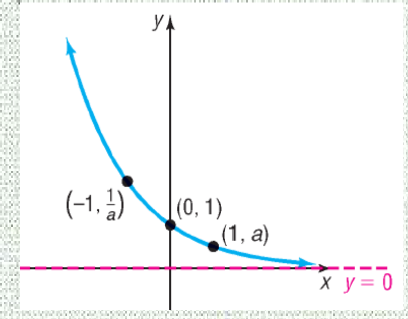
Domain: the interval $(-\infty, \infty)$; Range: the interval $(0, \infty)$

x-intercepts: none; y-intercept: 1

Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$

Increasing; one-to-one; smooth; continuous

$f(x) = a^x, \quad 0 < a < 1$



Domain: the interval $(-\infty, \infty)$; Range: the interval $(0, \infty)$.

x-intercepts: none; y-intercept: 1

Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$

Decreasing; one-to-one; smooth; continuous

If $a^u = a^v$, then $u = v$.