Section 6.4

Logarithmic Functions

DEFINITION

The **logarithmic function to the base *a***, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

EXAMPLE

Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$.

For example, $4 = \log_3 81$ is equivalent to $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$.

For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

OBJECTIVE 1

1 Change Exponential Expressions to Logarithmic Expressions and Logarithmic Expressions to Exponential Expressions

EXAMPLE

Changing Exponential Statements to Logarithmic Statements

Change each exponential expression to an equivalent expression involving a logarithm.

(a) $5^8 = t$ (b) $x^{-2} = 12$ (c) $e^x = 10$

EXAMPLE

Changing Logarithmic Statements to Exponential Statements

Change each logarithmic expression to an equivalent expression involving an exponent.

(a) $y = \log_2 21$ (b) $\log_z 12 = 6$ (c) $\log_2 10 = a$

OBJECTIVE 2

2 Evaluate Logarithmic Expressions

EXAMPLE

Finding the Exact Value of a Logarithmic Expression

$$(a)\ \log_3 81 \qquad (b)\ \log_2 \frac{1}{8}$$

OBJECTIVE 3

3 Determine the Domain of a Logarithmic Function

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

EXAMPLE

Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

$(a)\ f(x) = \log_3(x-2)$

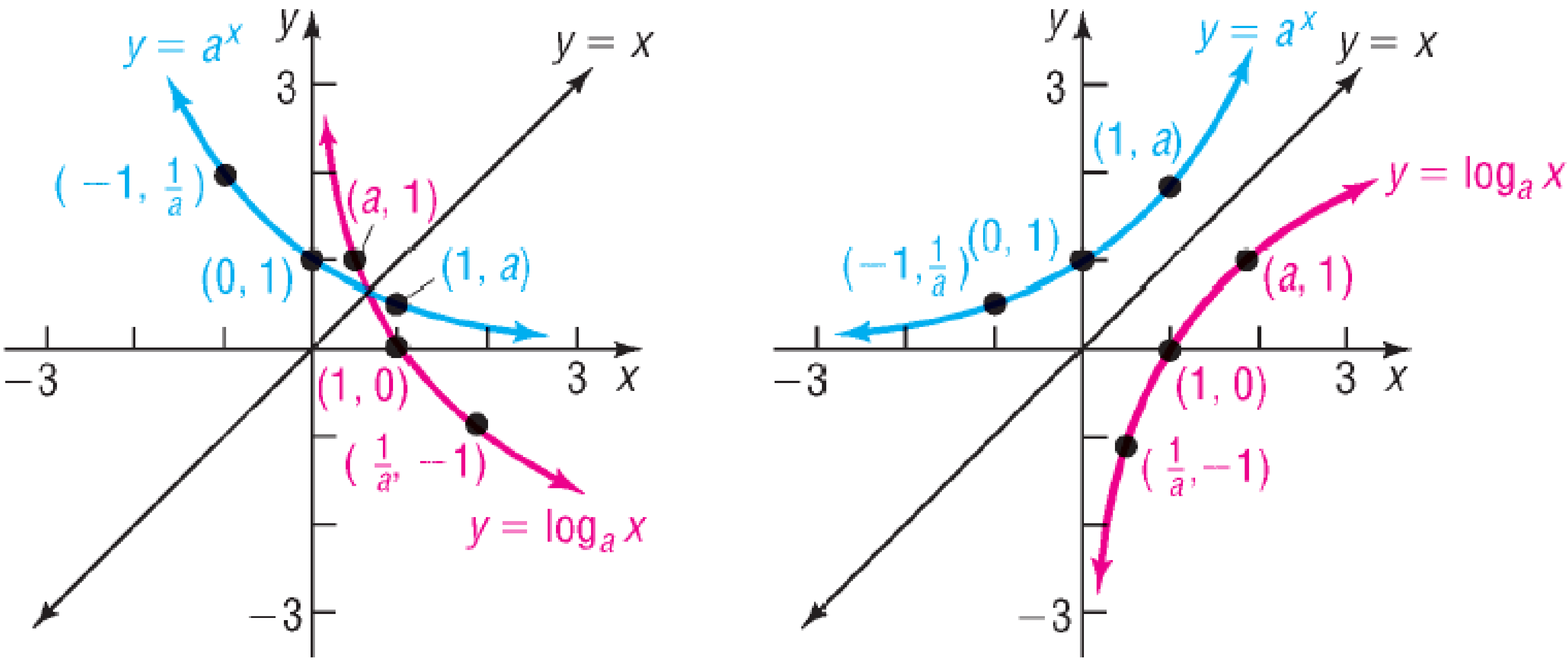
$(b)\ F(x) = \log_2\left(\frac{x+3}{x-1}\right)$

$(c)\ h(x) = \log_2|x-1|$

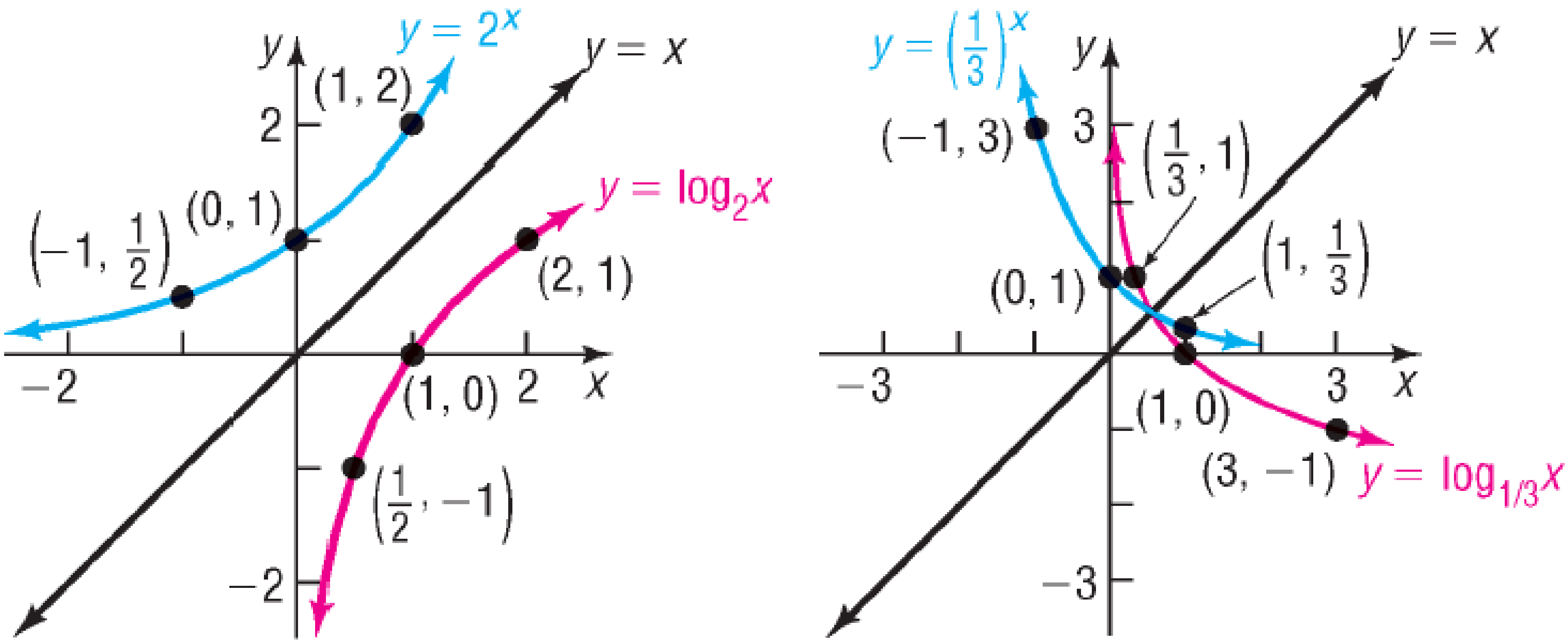
$(d)\ g(x) = \log_{\frac{1}{2}} x^2$

OBJECTIVE 4

4 Graph Logarithmic Functions


y = a^x
y
y = x
3
(−1, 1/a)
(a, 1)
(0, 1)
(1, a)
−3
3
x
(1, 0)
(1/a, −1)
y = log_a x
−3
y
y = a^x
y = x
3
(1, a)
y = log_a x
(−1, 1/a)
(0, 1)
(a, 1)
−3
(1, 0)
3
x
(1/a, −1)
−3

(a) $0 < a < 1$ **(b)** $a > 1$



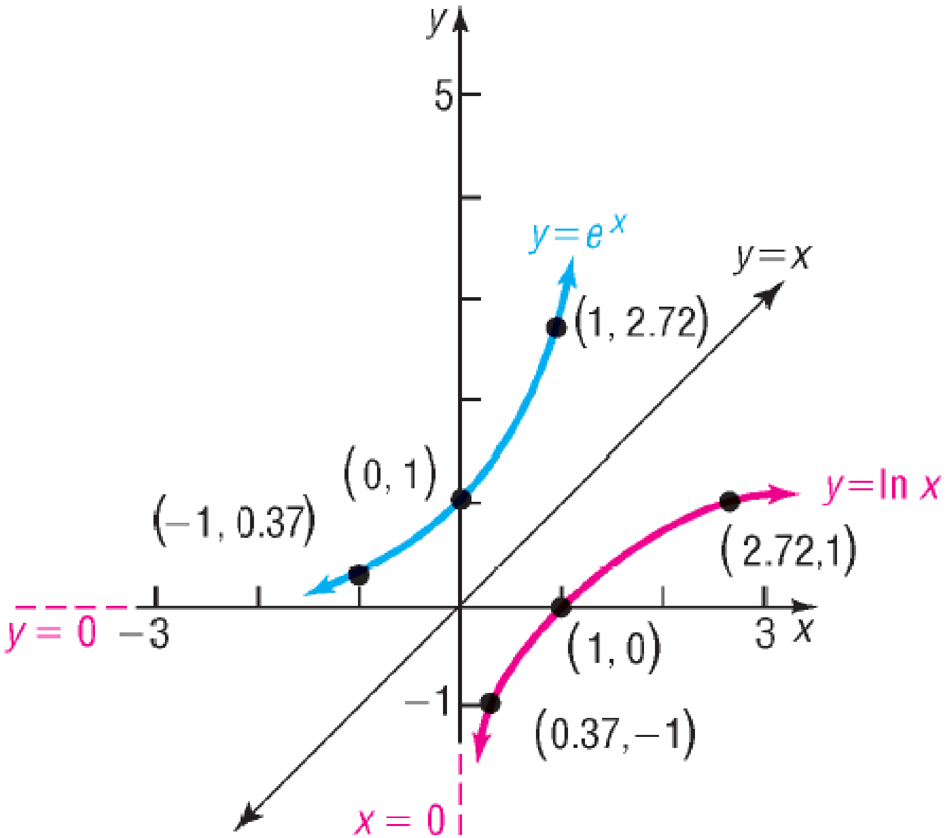
y = 2^x
y = x
y
(1, 2)
2
(0, 1)
(−1, 1/2)
y = log₂x
(2, 1)
−2
(1, 0)
2
x
(1/2, −1)
−2
y = (1/3)^x
y
y = x
(−1, 3)
3
(1/3, 1)
(1, 1/3)
(0, 1)
−3
3
x
(1, 0)
(3, −1)
y = log_{1/3}x
−3

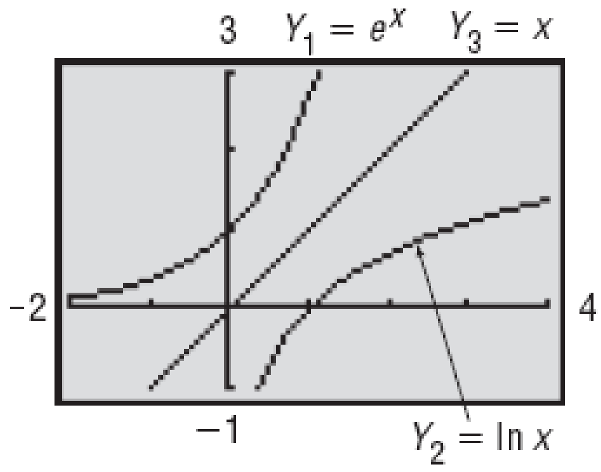
Properties of the Logarithmic Function $f(x) = \log_a x$

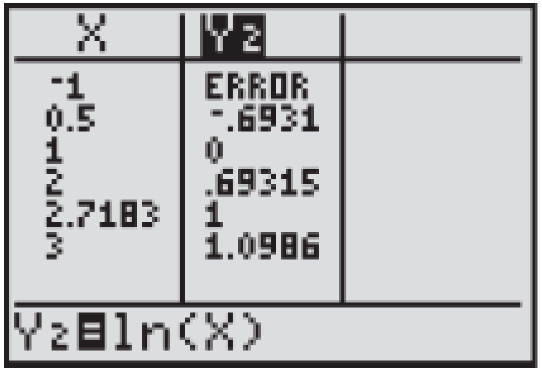
1. The domain is the set of positive real numbers; the range is the set of all real numbers.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis ($x = 0$) is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph is smooth and continuous, with no corners or gaps.

Natural Logarithm Function

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$







X	Y2
-1	ERROR
0.5	-.6931
1	0
2	.69315
2.7183	1
3	1.0986

Y2=ln(X)

EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\ln(x-1)$.

(b) Graph f.

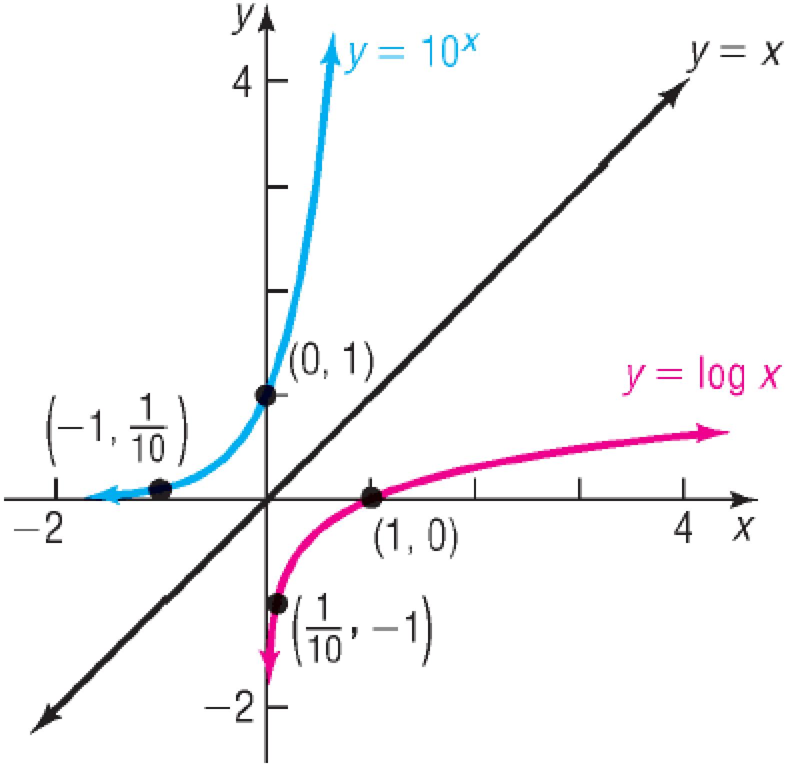
(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

$$y = \log x \quad \text{if and only if} \quad x = 10^y$$



EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -2\log(x+2)$

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

OBJECTIVE 5

5 Solve Logarithmic Equations

EXAMPLE

Solving a Logarithmic Equation

Solve: $(a)\ \log_2(2x+1)=3$ $(b)\ \log_x 343=3$

EXAMPLE

Using Logarithms to Solve Exponential Equations

Solve: $2e^{3x} = 6$

EXAMPLE Alcohol and Driving

The blood alcohol concentration (BAC) is the amount of alcohol in a person's bloodstream. A BAC of 0.04% means that a person has 4 parts alcohol per 10,000 parts blood in the body. Relative risk is defined as the likelihood of one event occurring divided by the likelihood of a second event occurring. For example, if an individual with a BAC of 0.02% is 1.4 times as likely to have a car accident as an individual who has not been drinking, the relative risk of an accident with a BAC of 0.02% is 1.4. Recent medical research suggests that the relative risk R of having an accident while driving a car can be modeled by the equation

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Research indicates that the relative risk of a person having an accident with a BAC of 0.02% is 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17%?

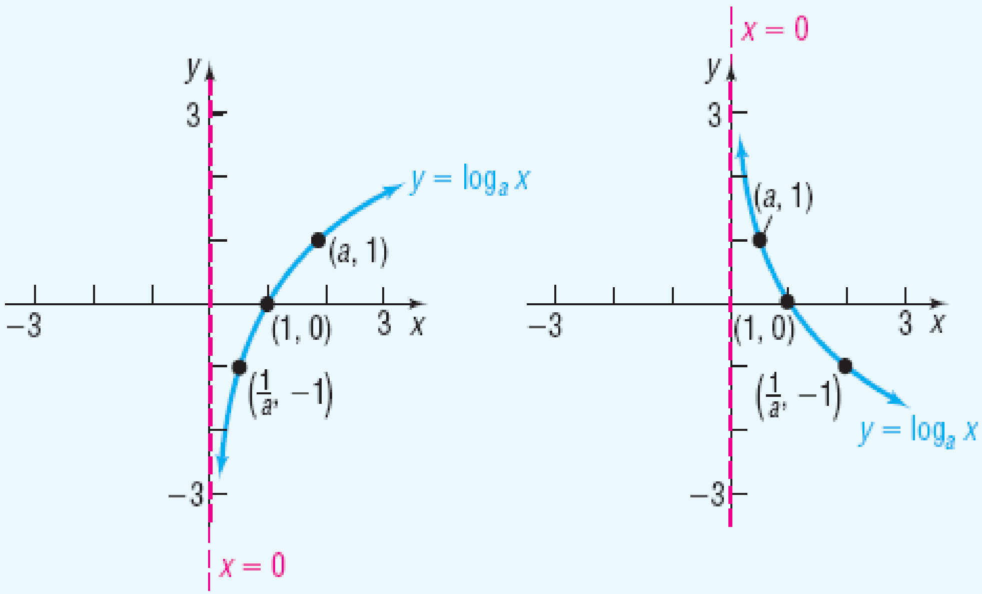
(c) Using this same value of k, what BAC corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI (driving under the influence)?

SUMMARY Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ ($y = \log_a x$ means $x = a^y$)	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one See Figure 44(a) for a typical graph.
$f(x) = \log_a x, \quad 0 < a < 1$ ($y = \log_a x$ means $x = a^y$)	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one See Figure 44(b) for a typical graph.

Figure 44



(a) $a > 1$ **(b)** $0 < a < 1$