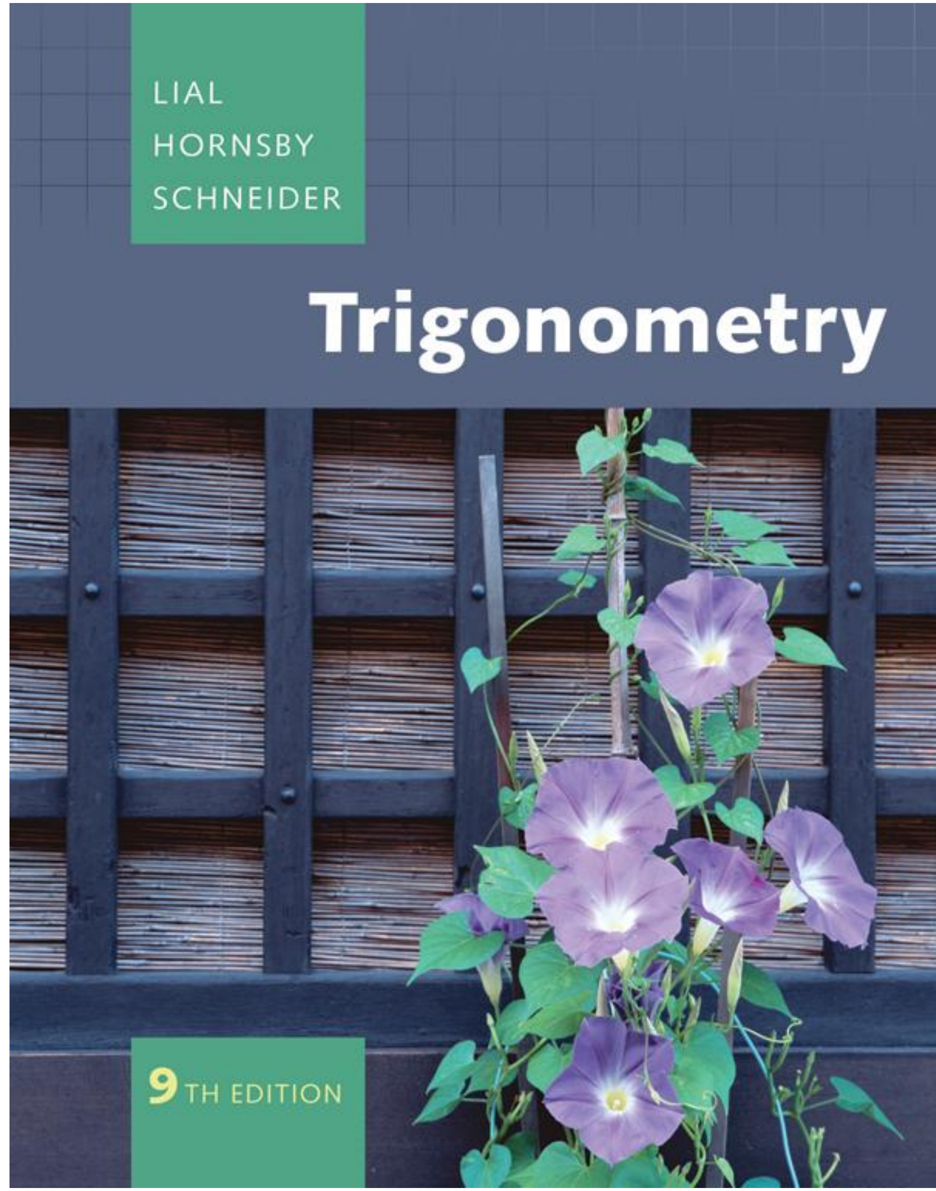
3

Radian Measure and Circular Functions



PEARSON Education

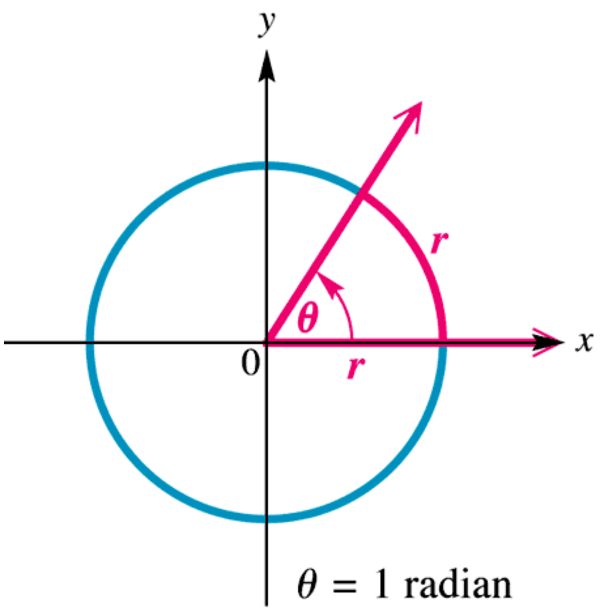
3 Radian Measure and Circular Functions

3.1 Radian Measure

Radian Measure   Converting Between Degrees and Radians
Finding Function Values for Angles in Radians

Radian

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of 1 radian.



Converting Between Degrees and Radians

Multiply a degree measure by $\frac{\pi}{180}$ radian and simplify to convert to radians.

Multiply a radian measure by $\frac{180°}{\pi}$ radian and simplify to convert to radians.

Example 1 CONVERTING DEGREES TO RADIANS

Convert each degree measure to radians.

(a) $45^\circ = 45\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{4}\text{ radian}$

(b) $-270^\circ = -270\left(\frac{\pi}{180}\text{ radian}\right) = -\frac{3\pi}{2}\text{ radian}$

(c) $249.8^\circ = 249.8\left(\frac{\pi}{180}\text{ radian}\right) \approx 4.360\text{ radians}$

Example 2 CONVERTING RADIANS TO DEGREES

Convert each radian measure to degrees.

(a) $$\frac{9\pi}{4} = \frac{9\pi}{4}\left(\frac{180°}{\pi}\right) = 405°$$

(b) $$-\frac{5\pi}{6} = -\frac{5\pi}{6}\left(\frac{180°}{\pi}\right) = -150°$$

(c) $$4.25 = 4.25\left(\frac{180°}{\pi}\right) \approx 253.5° = 243°30'$$

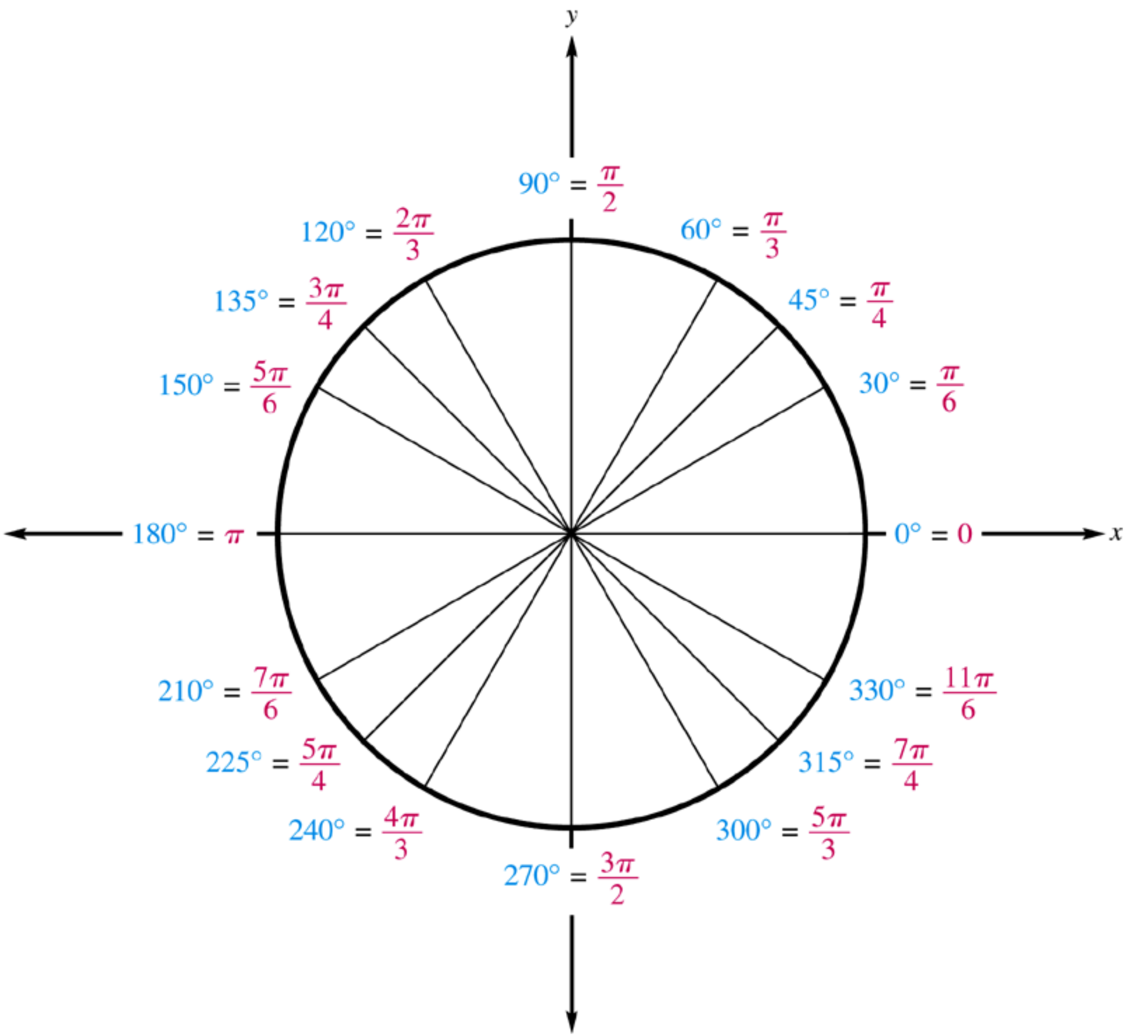
Agreement on Angle Measurement Units

If no unit of angle measure is specified, then the angle is understood to be measured in radians.

Equivalent Angle Measures in Degrees and Radians

Degrees	Radians		Degrees	Radians	
	Exact	Approximate		Exact	Approximate
$0°$	0	0	$90°$	$\frac{\pi}{2}$	1.57
$30°$	$\frac{\pi}{6}$	$.52$	$180°$	π	3.14
$45°$	$\frac{\pi}{4}$	$.79$	$270°$	$\frac{3\pi}{2}$	4.71
$60°$	$\frac{\pi}{3}$	1.05	$360°$	2π	6.28

Equivalent Angle Measures in Degrees and Radians



Example 3 FINDING FUNCTION VALUES OF ANGLES IN RADIAN MEASURE

Find each function value.

(a) $\tan\frac{2\pi}{3} = \tan\left(\frac{2\pi}{3}\cdot\frac{180°}{\pi}\right) = \tan 120° = -\sqrt{3}$

(b) $\sin\frac{3\pi}{2} = \sin 270° = -1$

(c) $\cos\left(-\frac{4\pi}{3}\right) = \cos\left(-\frac{4\pi}{3}\cdot\frac{180°}{\pi}\right) = \cos(-240°)$

$$= -\cos 60° = -\frac{1}{2}$$

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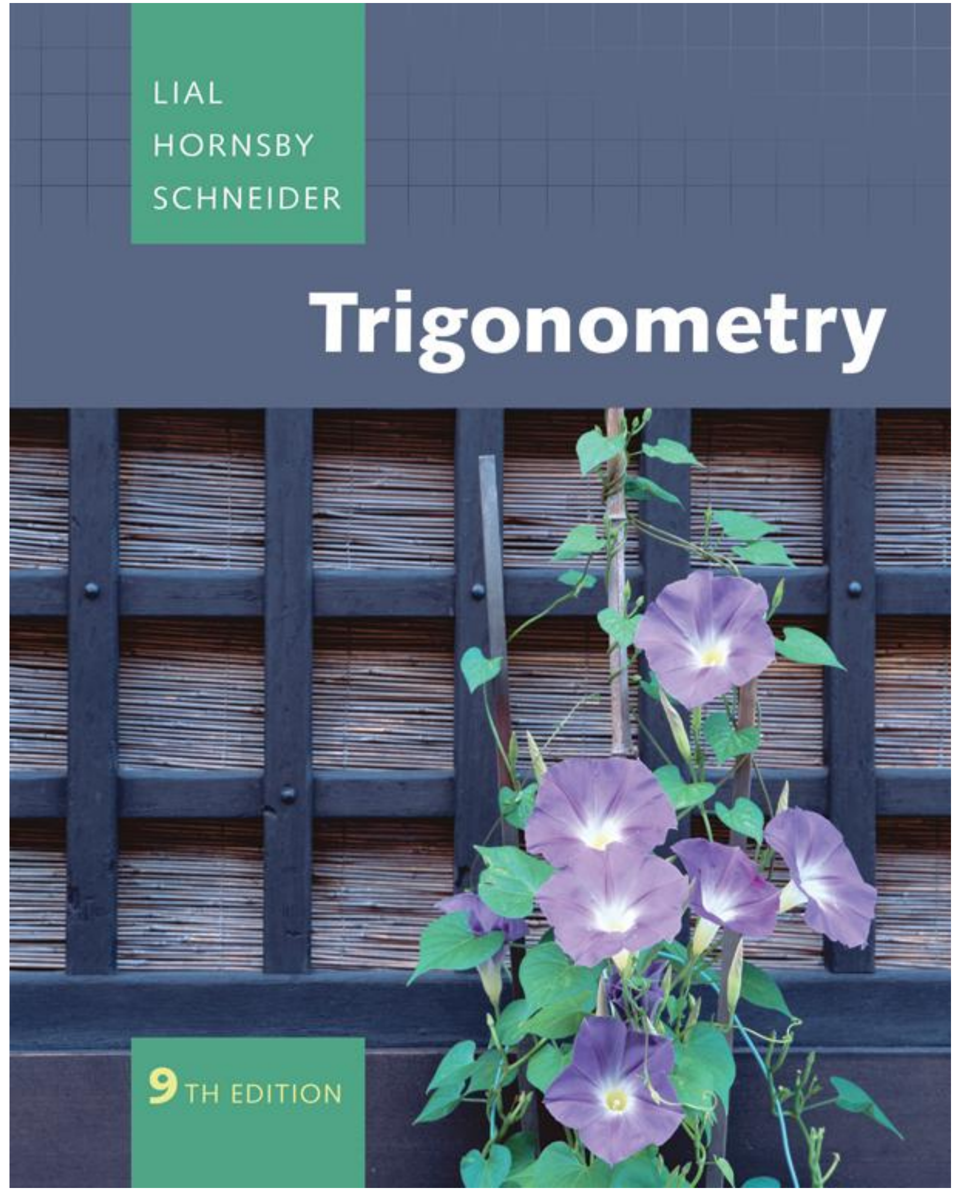
Radian Measure and Circular Functions



3 Radian Measure and Circular Functions

3.1 Radian Measure

3.2 Applications of Radian Measure

3.3 The Unit Circle and Circular Functions

3.4 Linear and Angular Speed

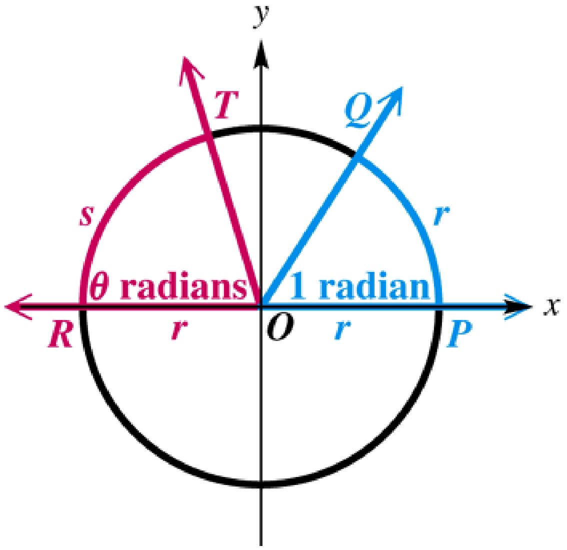
3.2 Applications of Radian Measure

Arc Length on a Circle   Area of a Sector of a Circle

Arc Length

The length *s* of the arc intercepted on a circle of radius *r* by a central angle of measure *θ* radians is given by the product of the radius and the radian measure of the angle, or

$$\boldsymbol{s = r\theta, \quad \theta \text{ in radians}}$$



Caution

Avoid the common error of applying this formula with θ in degree mode.

When applying the formula $s = r\theta$, the value of θ must be in radian mode.

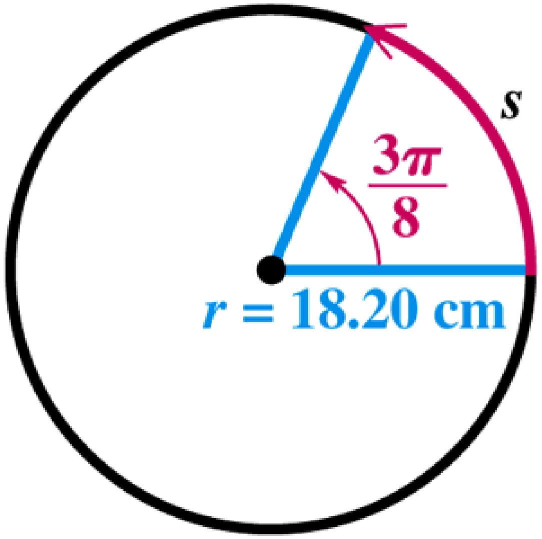
Example 1(a) FINDING ARC LENGTH USING $s = r\theta$

A circle has radius 18.20 cm. Find the length of the arc intercepted by a central angle with measure $\frac{3\pi}{8}$ radians.

$$s = r\theta$$

$$s = 18.20\left(\frac{3\pi}{8}\right) \text{ cm}$$

$$s \approx 21.44 \text{ cm}$$



Example 1(b) FINDING ARC LENGTH USING $s = r\theta$

A circle has radius 18.20 cm. Find the length of the arc intercepted by a central angle with measure 144 .

Convert θ to radians.

$$144^\circ = 144\left(\frac{\pi}{180}\right)$$

$$= \frac{4\pi}{5} \text{ radians}$$

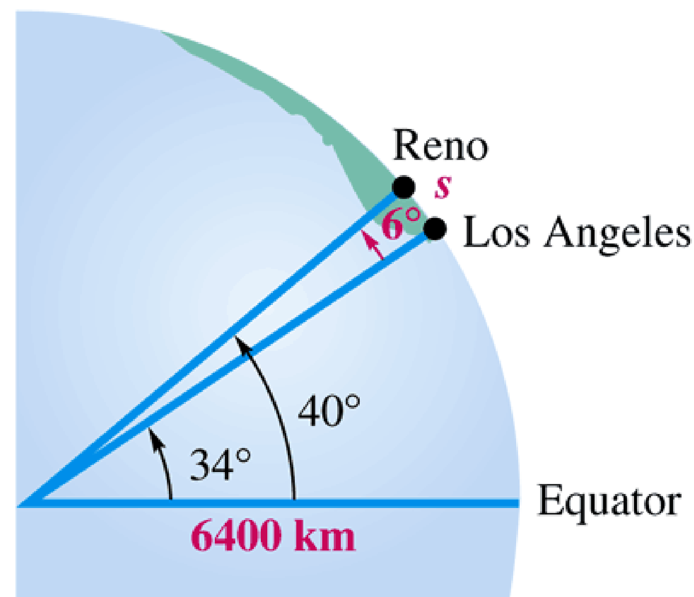
$$s = r\theta$$

$$s = 18.20\left(\frac{4\pi}{5}\right)$$

$$\approx 45.74 \text{ cm}$$

Example 2 USING LATITUDES TO FIND THE DISTANCE BETWEEN TWO CITIES

Reno, Nevada is approximately due north of Los Angeles. The latitude of Reno is 40° N, while that of Los Angeles is 34° N. The radius of Earth is about 6400 km. Find the north-south distance between the two cities.



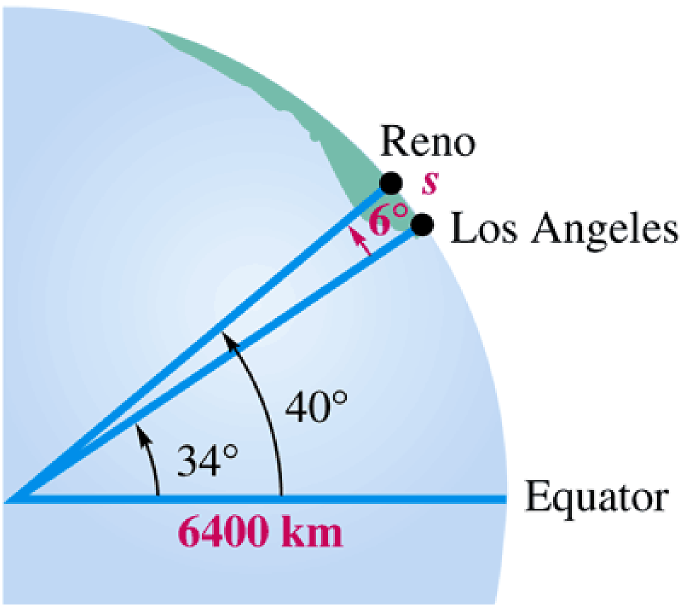
The central angle between Reno and Los Angeles is 40 – 34 = 6 . Convert 6 to radians:

$$6^\circ = 6\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ radian}$$

Example 2 USING LATITUDES TO FIND THE DISTANCE BETWEEN TWO CITIES (continued)

Use $s = r\theta$ to find the north-south distance between the two cities.

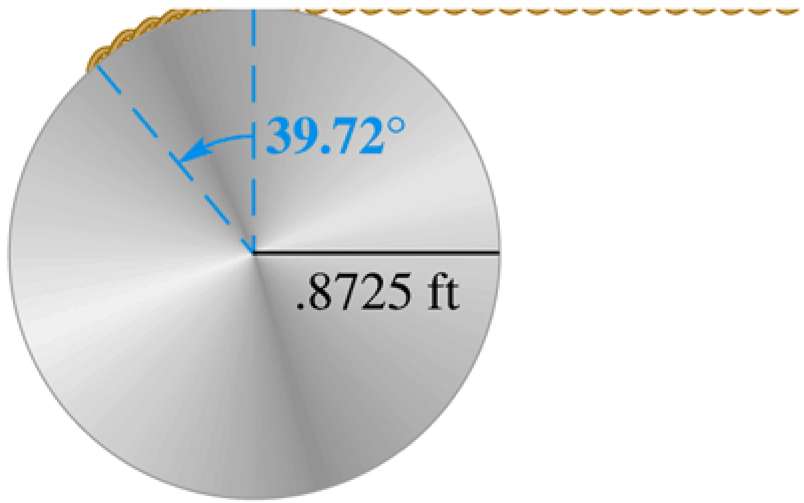
$$s = r\theta = 6400\left(\frac{\pi}{30}\right) \approx 670 \text{ km}$$



The north-south distance between Reno and Los Angeles is about 670 km.

Example 3 FINDING A LENGTH USING $s = r\theta$

A rope is being wound around a drum with radius .8725 ft. How much rope will be wound around the drum if the drum is rotated through an angle of 39.72 ?



The length of rope wound around the drum is the arc length for a circle of radius .8725 ft and a central angle of 39.72°.

Example 3 FINDING A LENGTH USING $s = r\theta$ (continued)

Use $s = r\theta$ to find the arc length, which is the length of the rope. Remember to convert 39.72   to radians

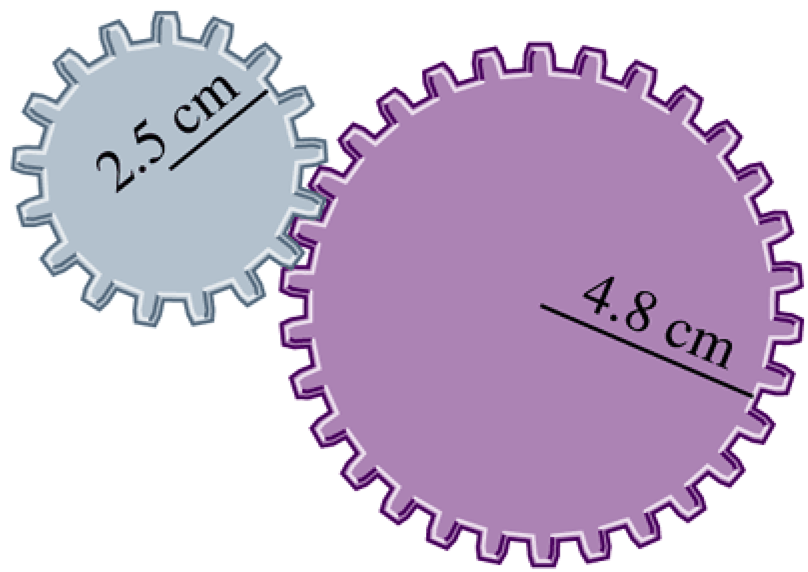
$$s = r\theta = .8725\left[39.72\left(\frac{\pi}{180}\right)\right] \approx .6049 \text{ ft}$$

The length of the rope wound around the drum is about .6049 ft.

▶ Example 4

FINDING AN ANGLE MEASURE USING $s = r\theta$

Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225 , through how many degrees will the larger gear rotate?



First find the radian measure of the angle, and then find the arc length on the smaller gear that determines the motion of the larger gear.

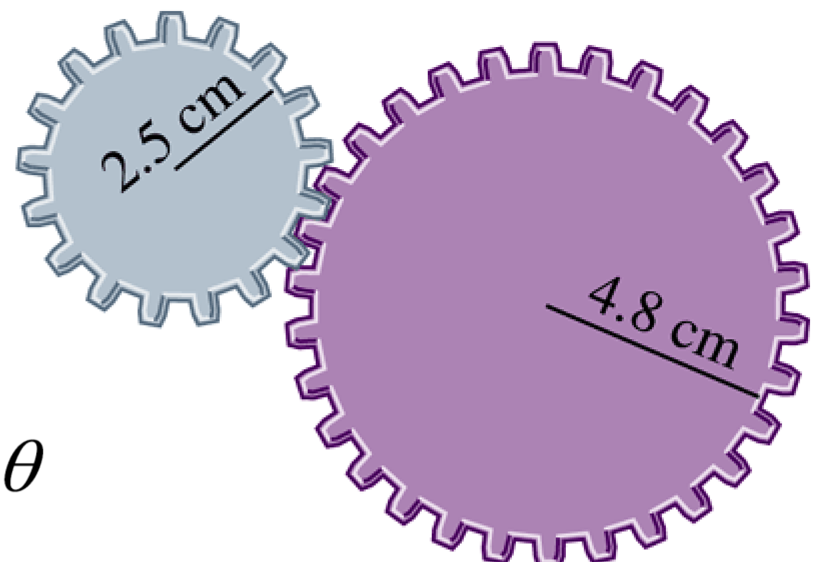
$$225° = \frac{5\pi}{4} \text{ radians}$$

Example 4 FINDING AN ANGLE MEASURE USING $s = r\theta$ (continued)

The arc length on the smaller gear is

$$s = r\theta = 2.5\left(\frac{5\pi}{4}\right)$$

$$= \frac{12.5\pi}{4} = \frac{25\pi}{8} \text{ cm}$$



An arc with this length on the larger gear corresponds to an angle measure θ:

$$s = r\theta$$

$$\frac{25\pi}{8} = 4.8\theta$$

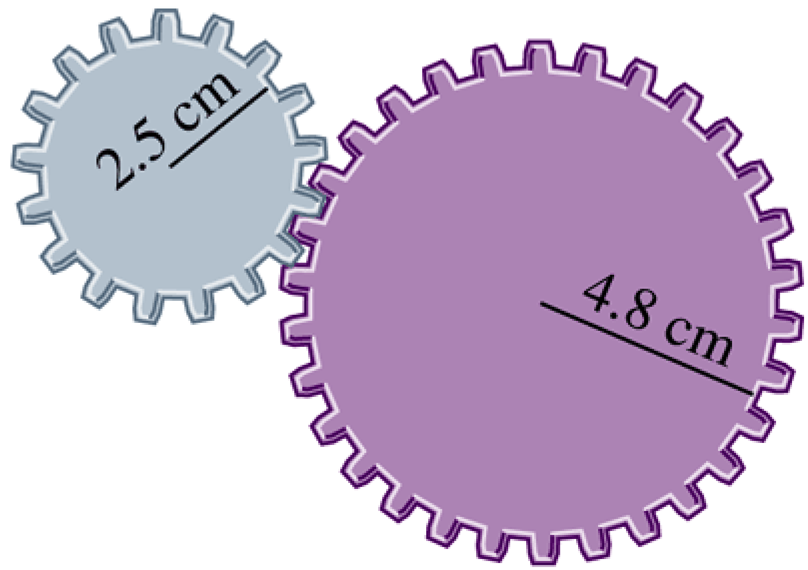
$$\frac{125\pi}{192} = \theta$$

Example 4 FINDING AN ANGLE MEASURE USING $s = r\theta$ (continued)

Convert θ to degrees:

$$\frac{125\pi}{192}\left(\frac{180°}{\pi}\right) \approx 117°$$

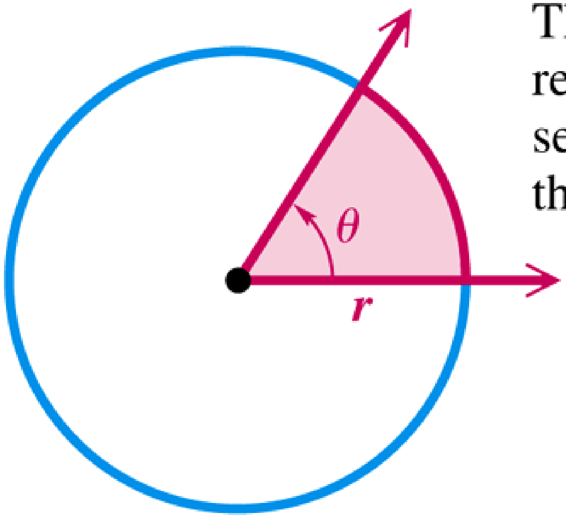
The larger gear rotates through an angle of 117 .



Area of a Sector of a Circle

A **sector of a circle** is a portion of the interior of a circle intercepted by a central angle.

Think of it as a “ piece of pie.”



The shaded region is a sector of the circle.

Area of a Sector

The area A of a sector of a circle of radius r and central angle θ is given by

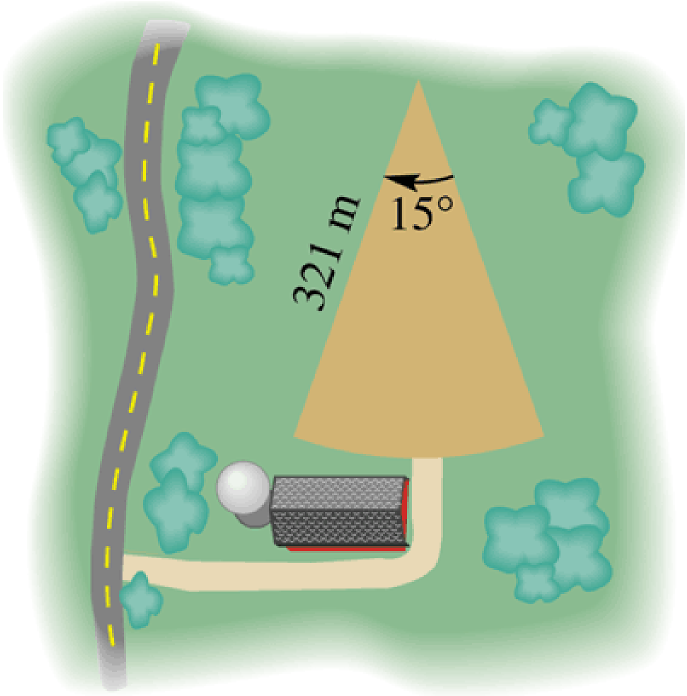
$$A = \frac{1}{2}r^2\theta, \quad \theta \text{ in radians}$$

Caution

The value of θ must be in radian mode when using the formula for the area of a sector.

Example 5 FINDING THE AREA OF A SECTOR-SHAPED FIELD

Find the area of the sector-shaped field shown in the figure.



$$A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \cdot 321^2 \cdot 15\left(\frac{\pi}{180}\right)$$

$$\approx 13{,}500 \text{ m}^2$$

Convert 15 to radians.

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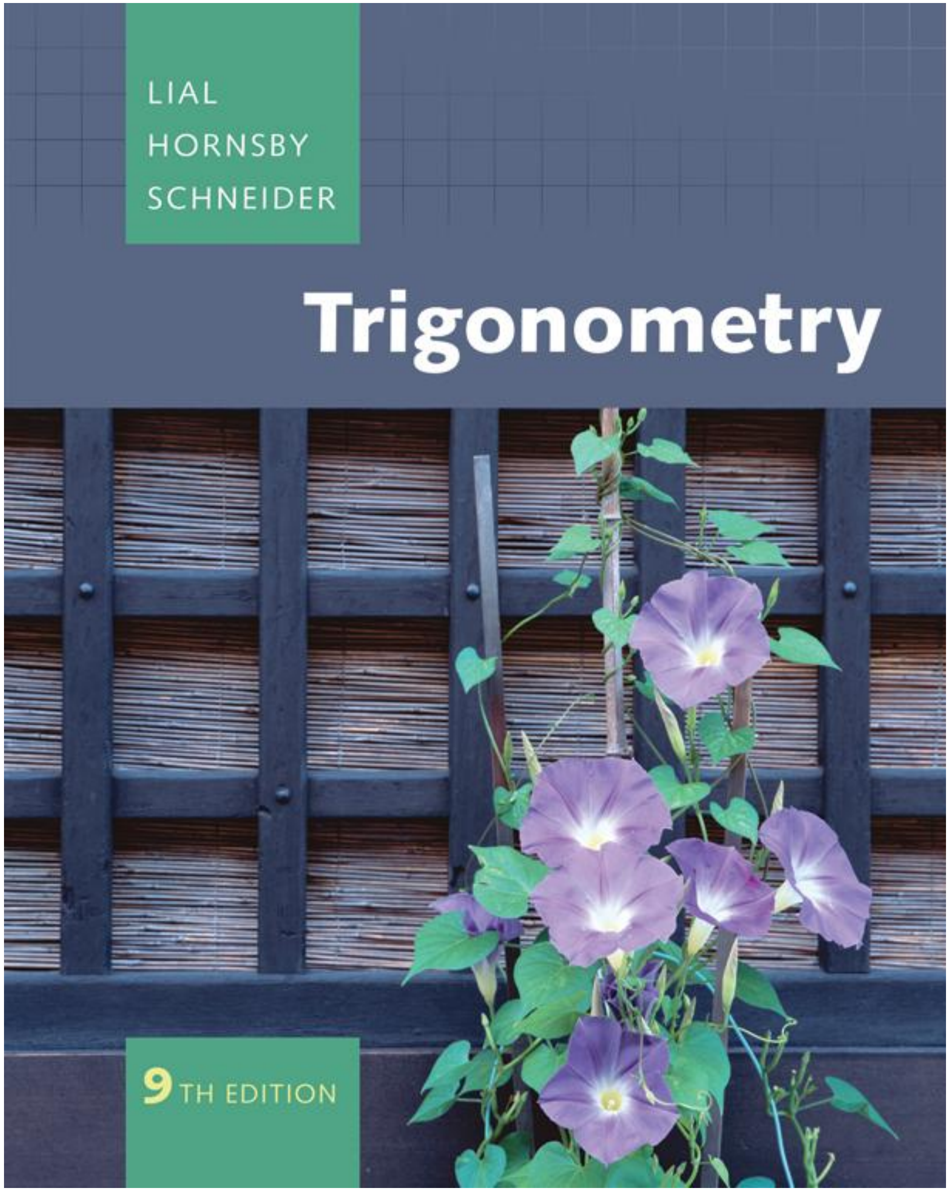
Radian Measure and Circular Functions



3 Radian Measure and Circular Functions

3.3 The Unit Circle and Circular Functions

Circular Functions   Finding Values of Circular Functions

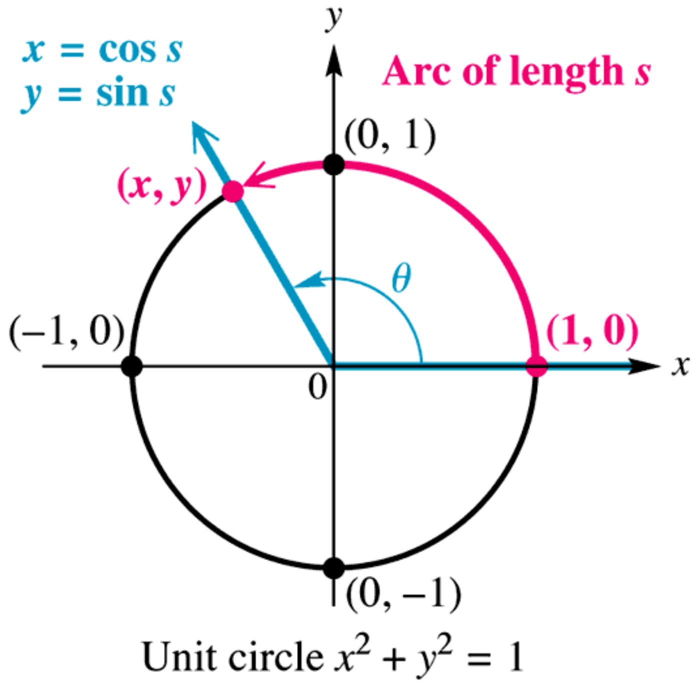
Determining a Number with a Given Circular Function Value

Applying Circular Functions

Circular Functions

A unit circle has its center at the origin and a radius of 1 unit.

The trigonometric functions of angle θ in radians are found by choosing a point (x, y) on the unit circle can be rewritten as functions of the arc length s.



When interpreted this way, they are called **circular functions.**

Circular Functions

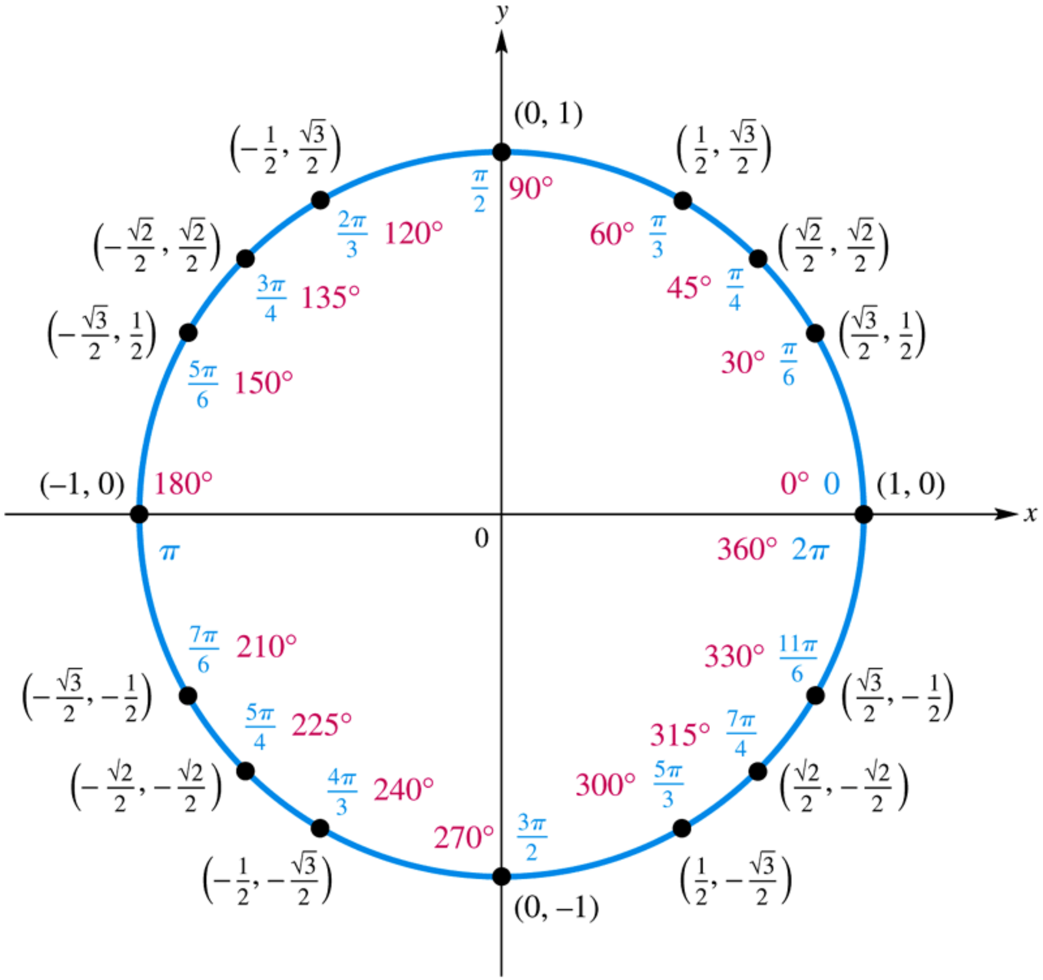
For any real number s represented by a directed arc on the unit circle,

$$\sin s = y \qquad \csc s = \frac{1}{y}\ (y \neq 0)$$

$$\cos s = x \qquad \sec s = \frac{1}{x}\ (x \neq 0)$$

$$\tan s = \frac{y}{x}\ (x \neq 0) \qquad \cot s = \frac{x}{y}\ (y \neq 0)$$

The Unit Circle



The unit circle $x^2 + y^2 = 1$

The Unit Circle

- The unit circle is symmetric with respect to the *x*-axis, the *y*-axis, and the origin.

  If a point (a, b) lies on the unit circle, so do $(a, –b)$, $(–a, b)$ and $(–a, –b)$.

The Unit Circle

- For a point on the unit circle, its **reference arc** is the shortest arc from the point itself to the nearest point on the *x*-axis.

For example, the quadrant I real number $\frac{\pi}{3}$ is associated with the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.

s	Quadrant of s	Symmetry Type and Corresponding Point	$\cos s$	$\sin s$
$\boldsymbol{\frac{\pi}{3}}$	I	not applicable; $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi - \frac{\pi}{3} = \boldsymbol{\frac{2\pi}{3}}$	II	y-axis; $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi + \frac{\pi}{3} = \boldsymbol{\frac{4\pi}{3}}$	III	origin; $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$2\pi - \frac{\pi}{3} = \boldsymbol{\frac{5\pi}{3}}$	IV	x-axis; $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

The Unit Circle

Since sin s = y and cos s = x, we can replace x and y in the equation of the unit circle

$$x^2 + y^2 = 1$$

to obtain the Pythagorean identity

$$\cos^2 s + \sin^2 s = 1.$$

Domains of Circular Functions

Sine and Cosine Functions: $(-\infty, \infty)$

Tangent and Secant Functions:

$$\left\{ s \mid s \neq (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer} \right\}$$

Cotangent and Cosecant Functions:

$$\left\{ s \mid s \neq n\pi, \text{ where } n \text{ is any integer} \right\}$$

Evaluating A Circular Function

Circular function values of real numbers are obtained in the same manner as trigonometric function values of angles measured in radians.

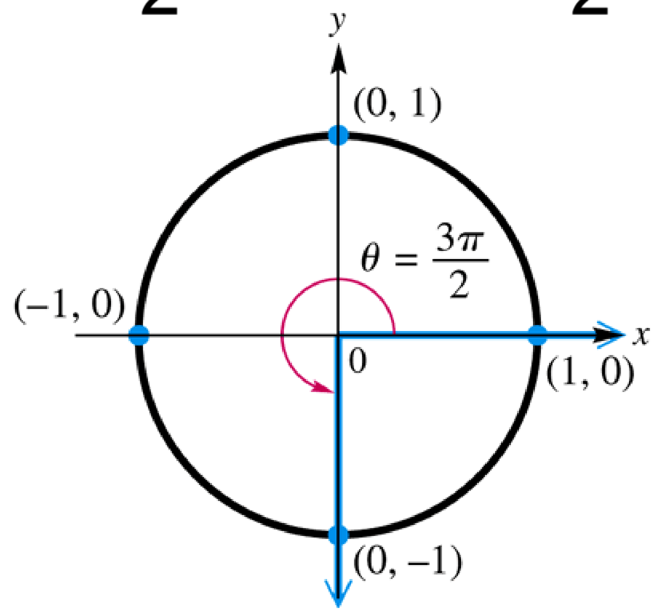
This applies both to methods of finding exact values (such as reference angle analysis) and to calculator approximations.

Calculators must be in radian mode when finding circular function values.

▶ Example 1 FINDING EXACT CIRCULAR FUNCTION VALUES

Find the exact values of $\sin\frac{3\pi}{2}$, $\cos\frac{3\pi}{2}$, and $\tan\frac{3\pi}{2}$.

Evaluating a circular function at the real number $\frac{3\pi}{2}$ is equivalent to evaluating it at $\frac{3\pi}{2}$ radians.



An angle of $\frac{3\pi}{2}$ intersects the circle at the point (0, –1).

Since $\sin s = y$, $\cos s = x$, and $\tan s = \frac{y}{x}$,

$\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, and $\tan\frac{3\pi}{2}$ is undefined.

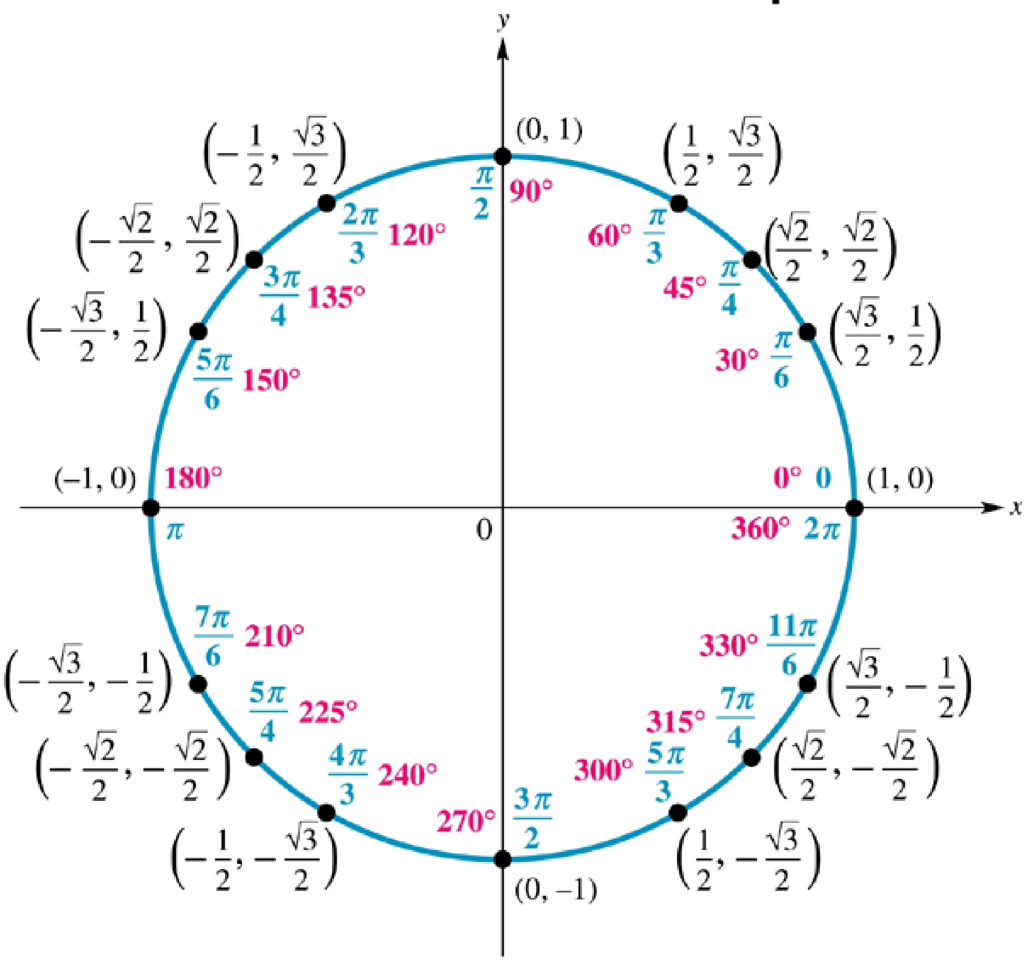
▶ Example 2(a) FINDING EXACT CIRCULAR FUNCTION VALUES

Use the figure to find the exact values of $\cos\dfrac{7\pi}{4}$ and $\sin\dfrac{7\pi}{4}$.

The real number $\dfrac{7\pi}{4}$ corresponds to the unit circle point $\left(\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2}\right)$.

$$\cos\frac{7\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\sin\frac{7\pi}{4}=-\frac{\sqrt{2}}{2}$$

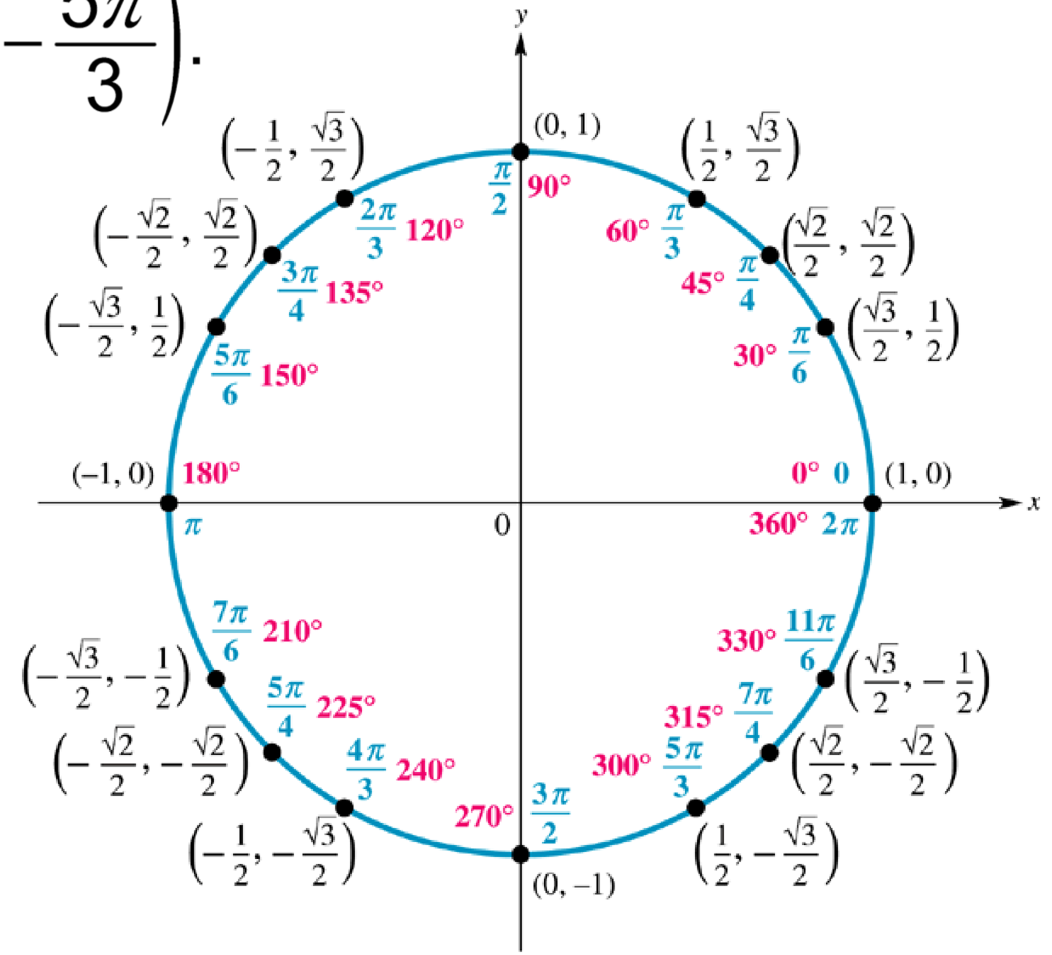


The unit circle $x^2 + y^2 = 1$

Example 2(b) FINDING EXACT CIRCULAR FUNCTION VALUES

Use the figure and the definition of tangent to find the exact value of $\tan\left(-\dfrac{5\pi}{3}\right)$.

Moving around the unit circle $\dfrac{5\pi}{3}$ units in the negative direction yields the same ending point as moving around the circle $\dfrac{\pi}{3}$ units in the positive direction.



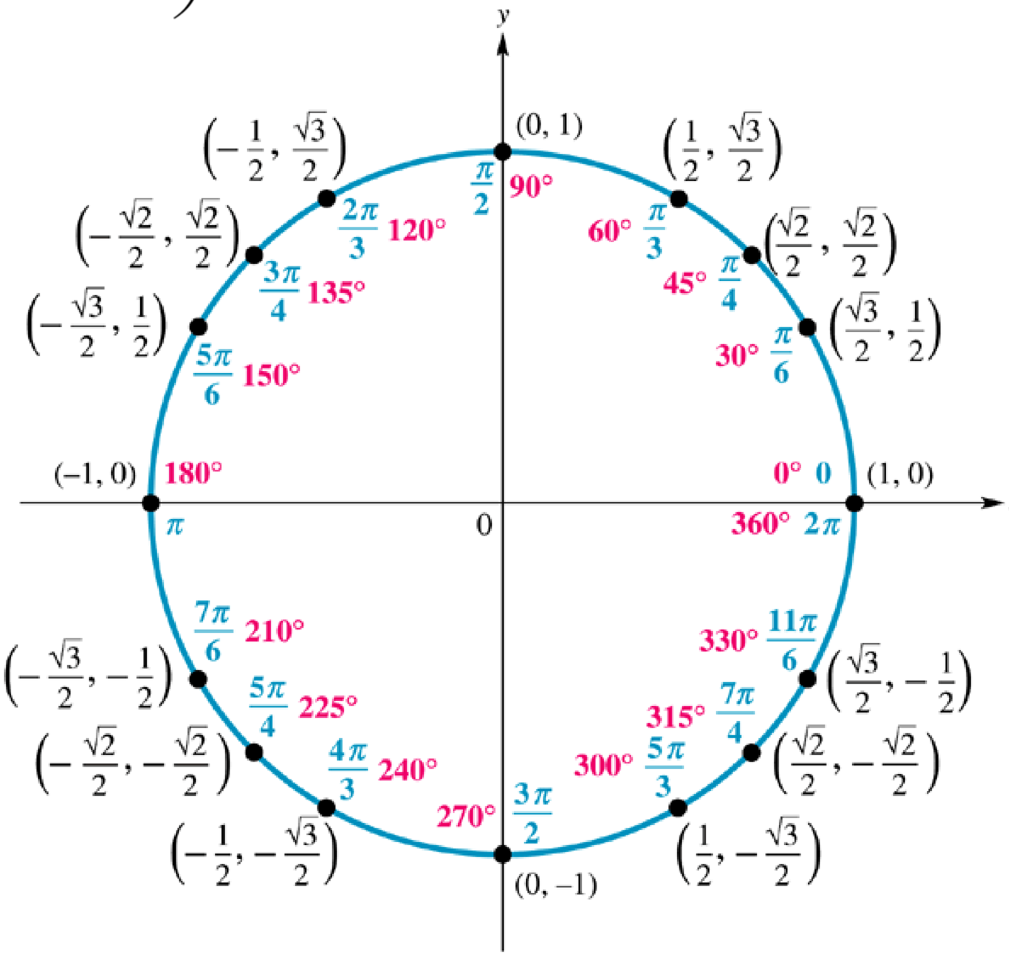
The unit circle $x^2 + y^2 = 1$

▶ Example 2(b) FINDING EXACT CIRCULAR FUNCTION VALUES

$-\dfrac{5\pi}{3}$ corresponds to $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

$$\boxed{\tan\theta = \frac{y}{x}}$$

$$\tan\left(-\frac{5\pi}{3}\right) = \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$



The unit circle $x^2 + y^2 = 1$

▶ Example 2(c) FINDING EXACT CIRCULAR FUNCTION VALUES

Use reference angles and degree/radian conversion to find the exact value of $\cos \frac{2\pi}{3}$.

An angle of $\frac{2\pi}{3}$ corresponds to an angle of 120 .

In standard position, 120 lies in quadrant II with a reference angle of 60 , so

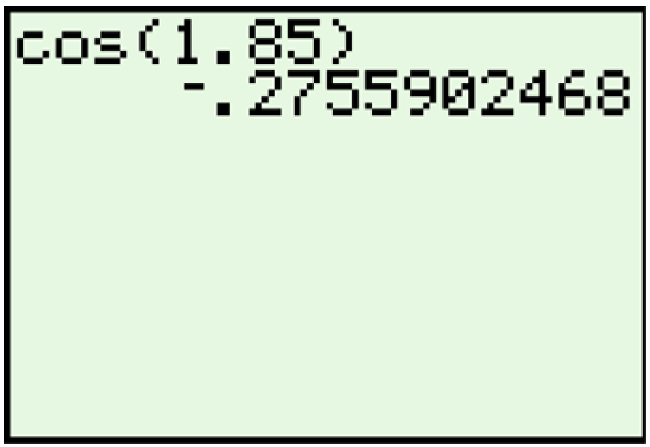
$$\cos \frac{2\pi}{3} = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$$

Cosine is negative in quadrant II.

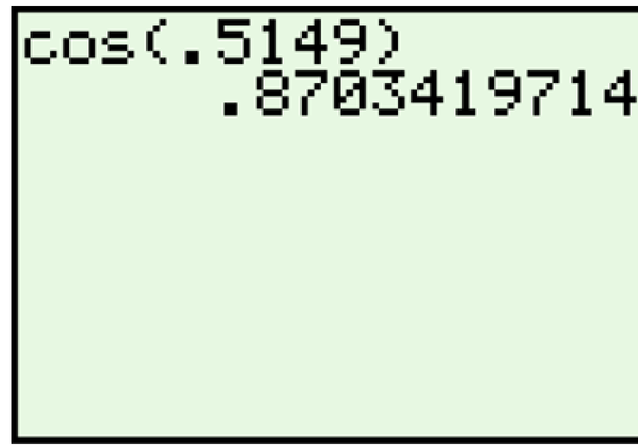
Example 3 APPROXIMATING CIRCULAR FUNCTION VALUES

Find a calculator approximation for each circular function value.

(a) cos 1.85 ≈ –.2756



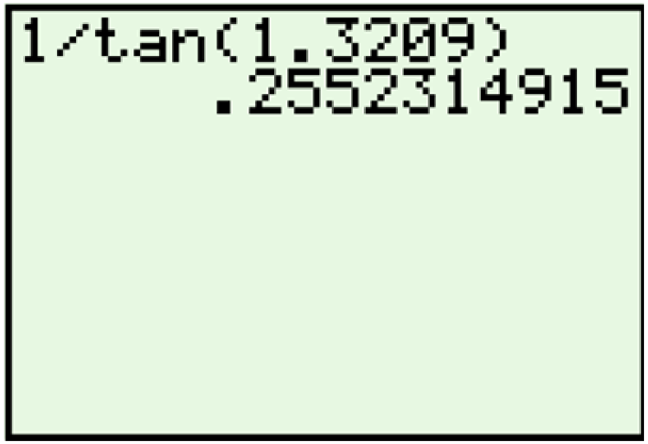
(b) cos .5149 ≈ .8703



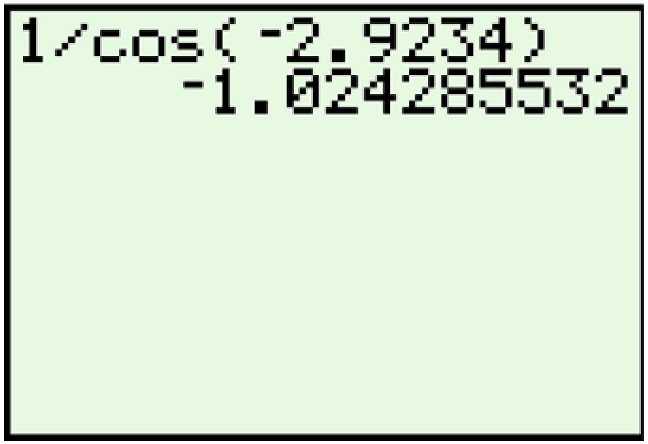
▶ Example 3 APPROXIMATING CIRCULAR FUNCTION VALUES (continued)

Find a calculator approximation for each circular function value.

(c) cot 1.3209 ≈ .2552



(d) sec –2.9234 ≈ –1.0243



Caution

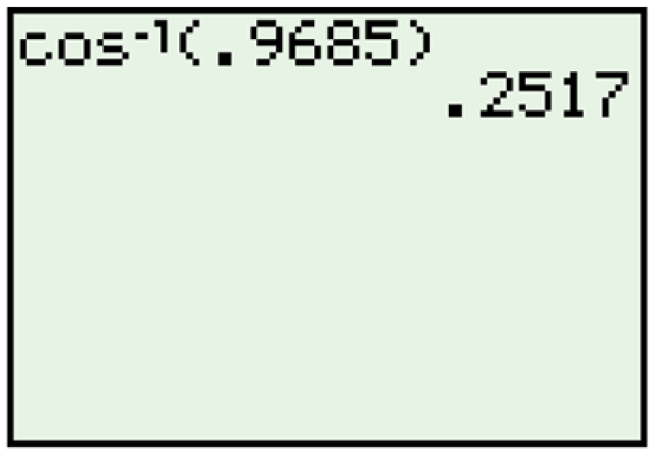
A common error in trigonometry is using a calculator in degree mode when radian mode should be used.

Remember, if you are finding a circular function value of a real number, the calculator must be in radian mode.

Example 4(a) FINDING A NUMBER GIVEN ITS CIRCULAR FUNCTION VALUE

Approximate the value of *s* in the interval $\left[0, \frac{\pi}{2}\right]$, if cos *s* = .9685.

Use the *inverse cosine* function of a calculator.

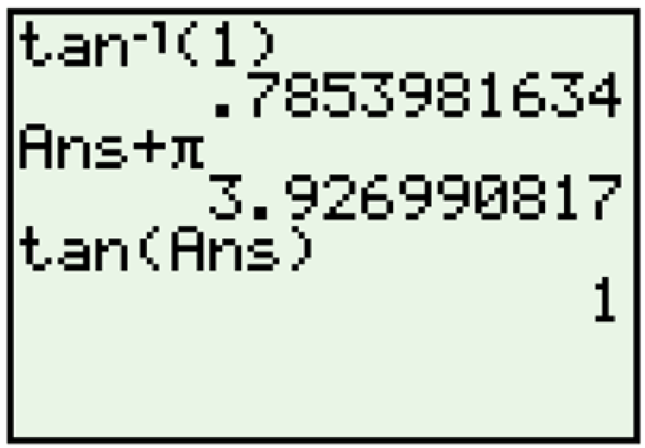


$\frac{\pi}{2} \approx 1.5708$, so in the given interval, *s* ≈ .2517.

Example 4(b) FINDING A NUMBER GIVEN ITS CIRCULAR FUNCTION VALUE

Find the exact value of *s* in the interval $\left[\pi, \dfrac{3\pi}{2}\right]$, if tan *s* = 1.

Recall that $\tan\dfrac{\pi}{4} = 1$, and in quadrant III, tan *s* is negative.



$$\tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{5\pi}{4} = 1, \text{ so } s = \frac{5\pi}{4}.$$

▶ Example 5 MODELING THE ANGLE OF ELEVATION OF THE SUN

The angle of elevation of the sun in the sky at any latitude *L* is calculated with the formula

$$\sin\theta = \cos D \cos L \cos\omega + \sin D \sin L$$

where $\theta = 0$ corresponds to sunrise and $\theta = \frac{\pi}{2}$ occurs if the sun is directly overhead. ω is the number of radians that Earth has rotated through since noon, when $\omega = 0$. *D* is the declination of the sun, which varies because Earth is tilted on its axis.

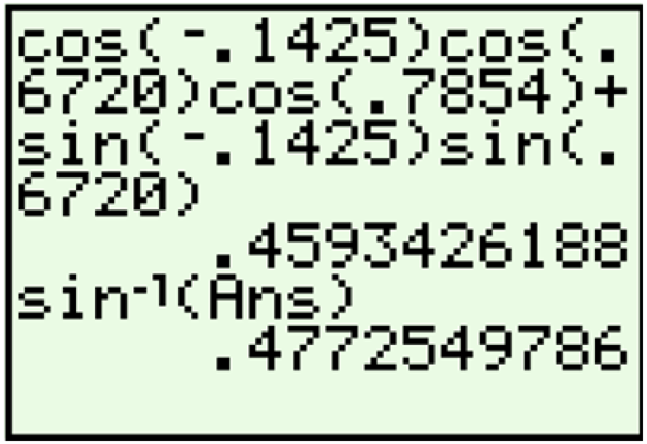
Example 5 MODELING THE ANGLE OF ELEVATION OF THE SUN (continued)

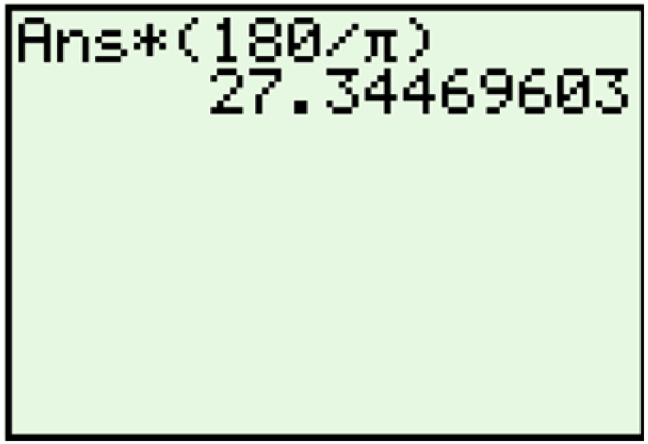
Sacramento, CA has latitude L = 38.5 or .6720 radian. Find the angle of elevation of the sun θ at 3 P.M. on February 29, 2008, where at that time, $D \approx -.1425$ and $\omega \approx .7854$.

$$\begin{aligned}
\sin\theta &= \cos D \cos L \cos\omega + \sin D \sin L \\
&= \cos(-.1425)\cos(.6720)\cos(.7854) \\
&\qquad + \sin(-.1425)\sin(.6720) \\
&\approx .4593426188
\end{aligned}$$

Example 5 MODELING THE ANGLE OF ELEVATION OF THE SUN (continued)

$$\sin \theta \approx .4593426188 \Rightarrow \theta \approx .4773 \text{ radian}$$





The angle of elevation of the sun is about .4773 radian or 27.3 .

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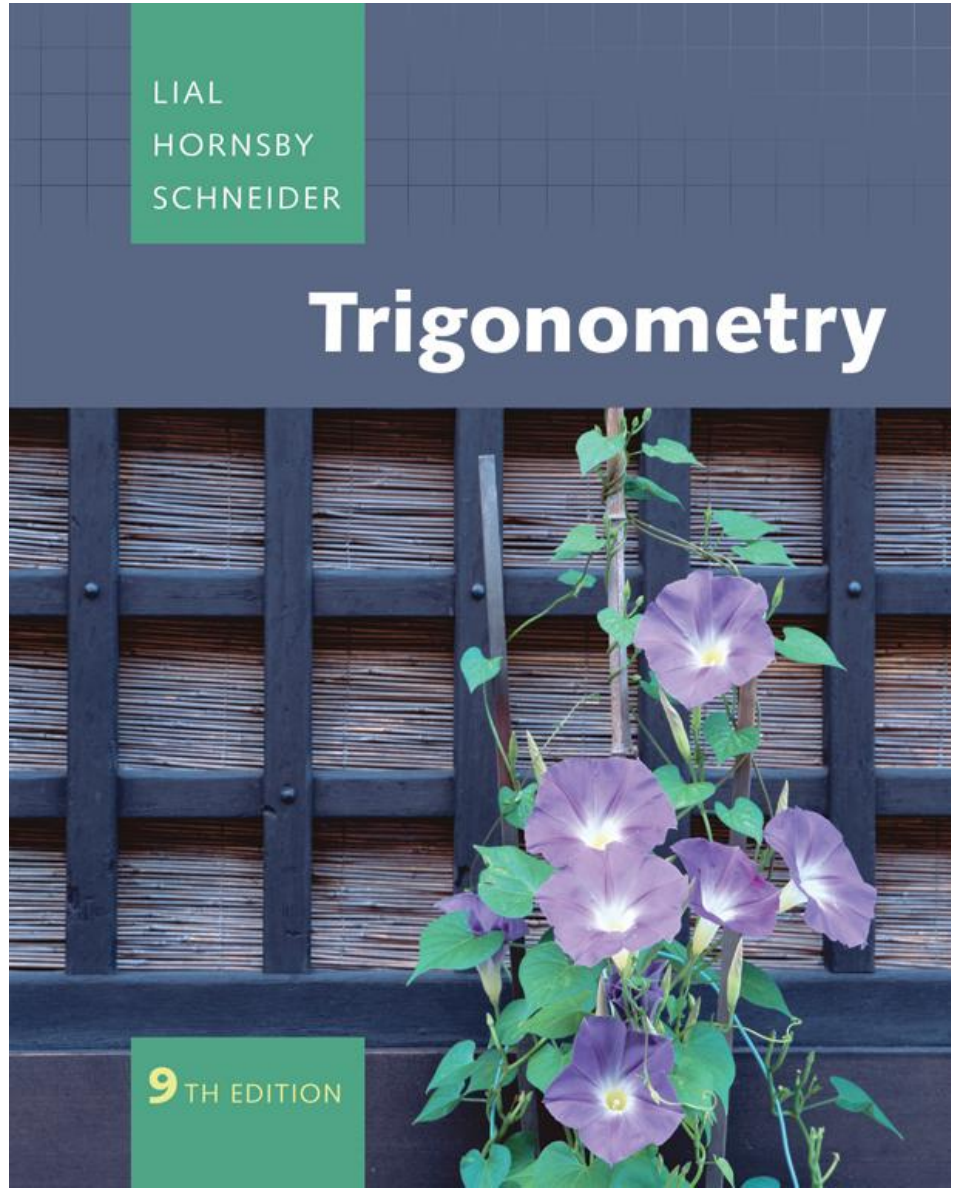
Radian Measure and Circular Functions



3 Radian Measure and Circular Functions

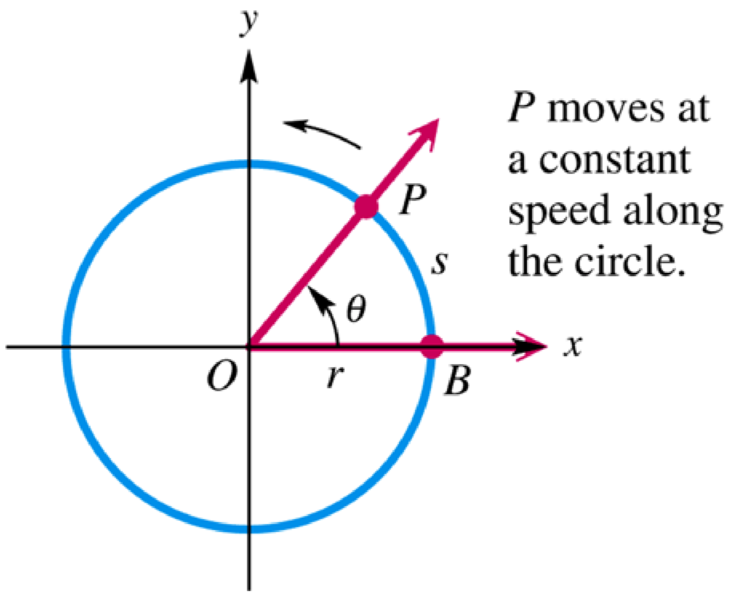
3.4 Linear and Angular Speed

Linear Speed ▪ Angular Speed

Linear Speed

Given a point *P* that moves at a constant speed along a circle of radius *r* and center *O.*

The measure of how fast the position of *P* is changing is its **linear speed**.



$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{or} \quad v = \frac{s}{t}$$

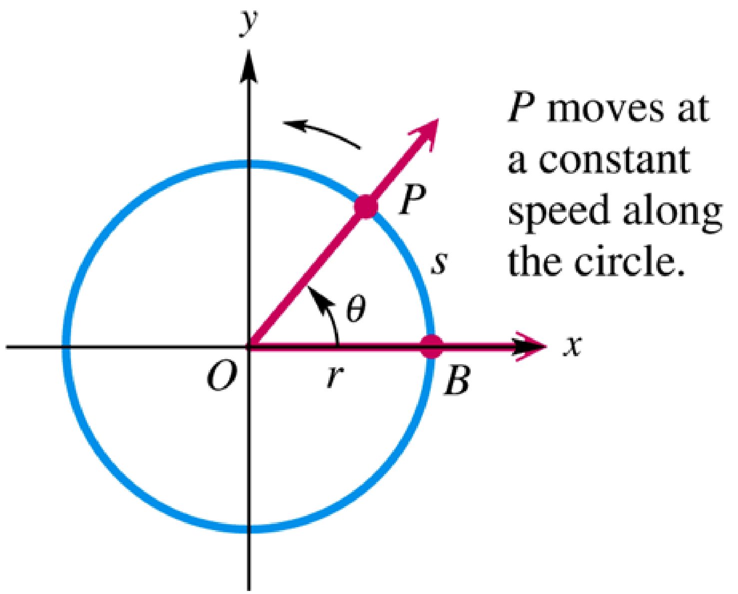
v is the linear speed, *s* is the length of the arc traced by point *P* at time *t*

Angular Speed

As point P moves along the circle, ray OP rotates about the origin.

The measure of how fast angle POB is changing is its **angular speed**.

$$\omega = \frac{\theta}{t}$$



ω is the angular speed, θ is the measure of angle POB (in radians) traced by point P at time t

Angular Speed	**Linear Speed**
$\omega = \dfrac{\theta}{t}$ (ω in radians per unit time, θ in radians)	$v = \dfrac{s}{t}$ $v = \dfrac{r\theta}{t}$ $v = r\omega$

Example 1 USING LINEAR AND ANGULAR SPEED FORMULAS

Suppose that *P* is on a circle with radius 10 cm, and ray *OP* is rotating with angular speed $\frac{\pi}{18}$ radian per second.

(a) Find the angle generated by *P* in 6 seconds.

$$\omega = \frac{\theta}{t} \Rightarrow \theta = \omega t$$

$$\theta = \frac{\pi}{18} \cdot 6 = \frac{\pi}{3} \text{ radians}$$

USING LINEAR AND ANGULAR SPEED FORMULAS (continued)

(b) Find the distance traveled by *P* along the circle in 6 seconds.

from part (a)

$$s = r\theta$$

$$s = 10\left(\frac{\pi}{3}\right) = \frac{10\pi}{3} \text{ cm}$$

(c) Find the linear speed of *P* in centimeters per second.

from part (b)

$$v = \frac{s}{t}$$

$$v = \frac{\frac{10\pi}{3}}{6} = \frac{10\pi}{18} = \frac{5\pi}{9} \text{ cm per sec}$$

Example 2 FINDING ANGULAR SPEED OF A PULLEY AND LINEAR SPEED OF A BELT

A belt runs a pulley of radius 6 cm at 80 revolutions per minute.

(a) Find the angular speed of the pulley in radians per second.

In one minute, the pulley makes 80 revolutions. Each revolution is 2π radians, so 80 revolutions = $80 \cdot 2\pi$ = 160π radians per minute.

There are 60 seconds in 1 minute, so

$$\omega = \frac{160\pi}{60} = \frac{8\pi}{3} \text{ radians per sec}$$

Example 2 FINDING ANGULAR SPEED OF A PULLEY AND LINEAR SPEED OF A BELT

(b) Find the linear speed of the belt in inches per second.

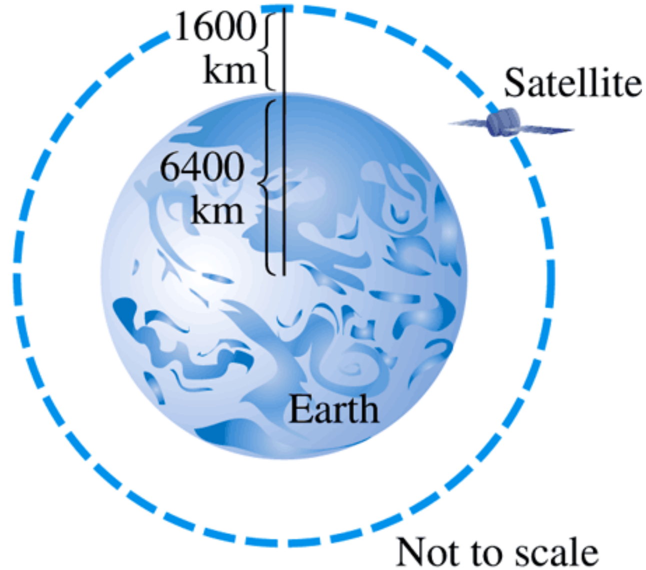
The linear speed of the belt is the same as that of a point on the circumference of the pulley.

$$v = r\omega = 6\left(\frac{8\pi}{3}\right) = 16\pi \approx 50 \text{ cm per sec}$$

from part (a)

▶ Example 3 FINDING LINEAR SPEED AND DISTANCE TRAVELED BY A SATELLITE

A satellite traveling in a circular orbit approximately 1600 km above the surface of Earth takes 2 hours to make an orbit. The radius of Earth is approximately 6400 km.



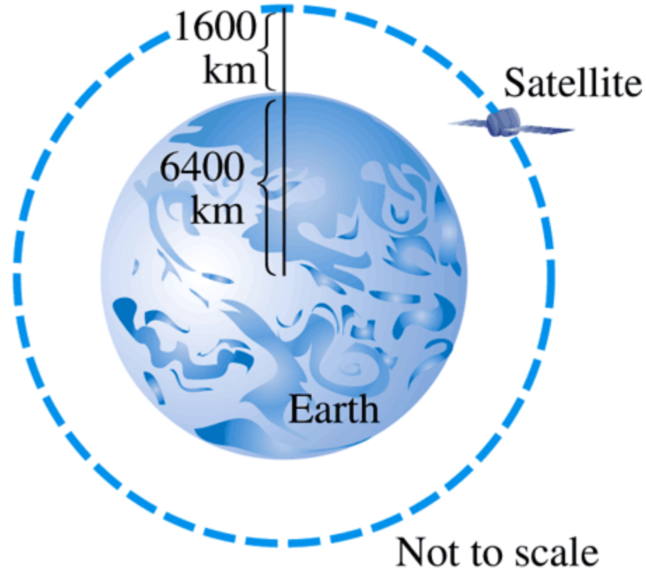
(a) Approximate the linear speed of the satellite in kilometers per hour.

The distance of the satellite from the center of Earth is approximately $r = 1800 + 6400 = 8200$ km.

Example 3 FINDING LINEAR SPEED AND DISTANCE TRAVELED BY A SATELLITE (continued)

For one orbit, $\theta = 2\pi$, so

$$s = r\theta = 8000(2\pi) = 16{,}000\pi \text{ km}$$



Since it takes 2 hours to complete an orbit, the linear speed is

$$v = \frac{s}{t} = \frac{16{,}000\pi}{2} = 8000\pi \approx 25{,}000 \text{ km per hr}$$

(b) Approximate the distance the satellite travels in 3.5 hours.

$$s = vt = 8000\pi(4.5) = 36{,}000\pi \approx 110{,}000 \text{ km}$$