4

Graphs of the Circular Functions
4.1 Graphs of the Sine and Cosine Functions

4.2 Translations of the Graphs of the Sine and Cosine Functions

4.3 Graphs of the Tangent and Cotangent Functions

4.4 Graphs of the Secant and Cosecant Functions

4.5 Harmonic Motion
4.1 Graphs of the Sine and Cosine Functions

- Periodic Functions
- Graph of the Sine Function
- Graph of the Cosine Function
- Graphing Techniques, Amplitude, and Period
- Using a Trigonometric Model
Periodic Functions

Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight.

This periodic graph represents a normal heartbeat.
**Periodic Function**

A periodic function is a function \( f \) such that

\[
f(x) = f(x + np),
\]

for every real number \( x \) in the domain of \( f \), every integer \( n \), and some positive real number \( p \). The least possible positive value of \( p \) is the period of the function.
The circumference of the unit circle is $2\pi$, so the least possible value of $p$ for which the sine and cosine functions repeat is $2\pi$.

*Therefore, the sine and cosine functions are periodic functions with period $2\pi$.***
### Values of the Sine and Cosine Functions

<table>
<thead>
<tr>
<th>As $s$ Increases from</th>
<th>$\sin s$</th>
<th>$\cos s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to $\frac{\pi}{2}$</td>
<td>Increases from 0 to 1</td>
<td>Decreases from 1 to 0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$ to $\pi$</td>
<td>Decreases from 1 to 0</td>
<td>Decreases from 0 to $-1$</td>
</tr>
<tr>
<td>$\pi$ to $\frac{3\pi}{2}$</td>
<td>Decreases from 0 to $-1$</td>
<td>Increases from $-1$ to 0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$ to $2\pi$</td>
<td>Increases from $-1$ to 0</td>
<td>Increases from 0 to 1</td>
</tr>
</tbody>
</table>
Sine Function \( f(x) = \sin x \)

Domain: \((-\infty, \infty)\)  
Range: \([-1,1]\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\pi/6)</td>
<td>(1/2)</td>
</tr>
<tr>
<td>(\pi/4)</td>
<td>(\sqrt{2}/2)</td>
</tr>
<tr>
<td>(\pi/3)</td>
<td>(\sqrt{3}/2)</td>
</tr>
<tr>
<td>(\pi/2)</td>
<td>1</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
</tr>
<tr>
<td>(3\pi/2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2\pi)</td>
<td>0</td>
</tr>
</tbody>
</table>

\(f(x) = \sin x, \ -2\pi \leq x \leq 2\pi\)
Sine Function  \( f(x) = \sin x \)

- The graph is continuous over its entire domain, \((-\infty, \infty)\).
- Its \(x\)-intercepts are of the form \(n\pi\), where \(n\) is an integer.
- Its period is \(2\pi\).
- The graph is symmetric with respect to the origin, so the function is an odd function. For all \(x\) in the domain, \(\sin(-x) = -\sin(x)\).
Cosine Function $f(x) = \cos x$

Domain: $(-\infty, \infty)$    Range: $[-1, 1]$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-1</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1</td>
</tr>
</tbody>
</table>

$f(x) = \cos x$, $-2\pi \leq x \leq 2\pi$
Cosine Function  $f(x) = \cos x$

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its $x$-intercepts are of the form $(2n + 1)\frac{\pi}{2}$, where $n$ is an integer.
- Its period is $2\pi$.
- The graph is symmetric with respect to the $y$-axis, so the function is an even function. For all $x$ in the domain, $\cos(-x) = \cos(x)$. 
Example 1

GRAPHING $y = a \sin x$

Graph $y = 2 \sin x$, and compare to the graph of $y = \sin x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$2 \sin x$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
</tbody>
</table>

The shape of the graph is the same as the shape of $y = \sin x$.

The range of $y = 2 \sin x$ is $[-2, 2]$. 
Example 1

GRAPHING $y = a \sin x$ (continued)

![Graph showing period of $2\pi$ for $y = 2 \sin x$ and $y = \sin x$]
Amplitude

The graph of \( y = a \sin x \) or \( y = a \cos x \), with \( a \neq 0 \), will have the same shape as the graph of \( y = \sin x \) or \( y = \cos x \), respectively, except with range \([-|a|, |a|]\). The amplitude is \(|a|\).
Graphs of the Sine and Cosine Functions

No matter what the value of the amplitude, the periods of $y = a \sin x$ and $y = a \cos x$ are still $2\pi$.

Now consider $y = \sin 2x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\pi$</th>
<th>$\frac{5\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{7\pi}{4}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
<td>$\frac{5\pi}{2}$</td>
<td>$6\pi$</td>
<td>$\frac{7\pi}{2}$</td>
<td>$4\pi$</td>
</tr>
<tr>
<td>$\sin 2x$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

One complete cycle occurs in $\pi$ units.
Graphs of the Sine and Cosine Functions

Now consider \( y = \sin 4x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( \frac{\pi}{8} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{3\pi}{8} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{5\pi}{8} )</th>
<th>( \frac{3\pi}{4} )</th>
<th>( \frac{7\pi}{8} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4x )</td>
<td>( 0 )</td>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
<td>( \frac{5\pi}{2} )</td>
<td>( 6\pi )</td>
<td>( \frac{7\pi}{2} )</td>
<td>( 4\pi )</td>
</tr>
<tr>
<td>( \sin 4x )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

One complete cycle occurs in \( \frac{\pi}{2} \) units.

In general, the graph of a function of the form \( y = \sin bx \) or \( y = \cos bx \), for \( b > 0 \), will have a period different from \( 2\pi \) when \( b \neq 1 \).
Period

For $b > 0$, the graph of $y = \sin bx$ will have the resemble that of $y = \sin x$, but with period $\frac{2\pi}{b}$.

For $b > 0$, the graph of $y = \cos bx$ will have the resemble that of $y = \cos x$, but with period $\frac{2\pi}{b}$. 
Divide the interval $\left[0, \frac{2\pi}{b}\right]$ into four equal parts to obtain the values for which $\sin bx$ or $\cos bx$ equal $-1$, $0$, or $1$.

These values give the minimum points, $x$-intercepts, and maximum points on the graph.

Find the midpoint of the interval by adding the $x$-values of the endpoints and dividing by 2. Then find the midpoints of the two intervals using the same procedure.
Example 2

**GRAPHING** \( y = \sin bx \)

Graph \( y = \sin 2x \) and compare to the graph of \( y = \sin x \).

The coefficient of \( x \) is 2, so \( b = 2 \), and the period is \( \frac{2\pi}{2} = \pi \).

The endpoints are 0 and \( \pi \), and the three points between the endpoints are

\[
\frac{1}{4}(0 + \pi), \quad \frac{1}{2}(0 + \pi), \quad \text{and} \quad \frac{3}{4}(0 + \pi).
\]
The $x$-values are

\[
\begin{array}{cccccccc}
0 & \frac{\pi}{4} & \frac{\pi}{2} & \frac{3\pi}{4} & \pi & \frac{5\pi}{4} & \frac{3\pi}{2} & \frac{7\pi}{4} & 2\pi \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\text{Left endpoint} & \text{First-quarter point} & \text{Midpoint} & \text{Third-quarter point} & \text{Right endpoint}
\end{array}
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\pi$</th>
<th>$\frac{5\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{7\pi}{4}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
<td>$\frac{5\pi}{2}$</td>
<td>$6\pi$</td>
<td>$\frac{7\pi}{2}$</td>
<td>$4\pi$</td>
</tr>
<tr>
<td>sin$2x$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 2

GRAPHING $y = \sin bx$ (continued)

$y = \sin x$

$y = \sin 2x$
Graph $y = \cos \frac{2}{3}x$ over one period.

The period is $\frac{2\pi}{\frac{2}{3}} = 2\pi \cdot \frac{3}{2} = 3\pi$.

The endpoints are 0 and $3\pi$, and the three points between the endpoints are $\frac{3\pi}{4}, \frac{3\pi}{2},$ and $\frac{9\pi}{4}$. 
Example 3

GRAPHING $y = \cos bx$ (continued)

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$3\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{3}x$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\cos \frac{2}{3}x$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The amplitude is 1.
Guidelines for Sketching Graphs of Sine and Cosine Functions

**Step 1** Find the period, $\frac{2\pi}{b}$. Start with 0 on the $x$-axis, and lay off a distance of $\frac{2\pi}{b}$.

**Step 2** Divide the interval into four equal parts.

**Step 3** Evaluate the function for each of the five $x$-values resulting from Step 2. The points will be maximum points, minimum points, and $x$-intercepts.
Step 4  Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5  Draw the graph over additional periods as needed.
Graph $y = -2 \sin 3x$ over one period.

**Step 1**

The coefficient of $x$ is 3, so $b = 3$, and the period is $\frac{2\pi}{3}$.

The function will be graphed over the interval $[0, \frac{2\pi}{3}]$.

**Step 2**

Divide the interval $[0, \frac{2\pi}{3}]$ into four equal parts to get the $x$-values $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and $\frac{2\pi}{3}$. 
**Example 4**

GRAPHING $y = a \sin bx$ (continued)

**Step 3**

Make a table of values determined by the $x$-values from Step 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\sin 3x$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$-2 \sin 3x$</td>
<td>0</td>
<td>$-2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
**Example 4**

**GRAPHING** \( y = a \sin bx \) (continued)

**Steps 4, 5**

Plot the points \((0,0), \left(\frac{\pi}{6}, -2\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{\pi}{2}, \right), \) and \(\left(\frac{2\pi}{3}, 0\right)\).

Join the points with a sinusoidal curve with amplitude 2. The graph can be extended by repeating the cycle.
Note

When $a$ is negative, the graph of $y = a \sin bx$ is the reflection across the x-axis of the graph of $y = |a| \sin bx$. 
Graphing $y = a \cos bx$ for $b$ equal to a multiple of $\pi$

Graph $y = -3 \cos \pi x$ over one period.

**Step 1**

Since $b = \pi$, the period is $\frac{2\pi}{\pi} = 2$.

The function will be graphed over the interval $[0, 2]$.

**Step 2**

Divide the interval $[0, 2]$ into four equal parts to get the $x$-values $0, \frac{1}{2}, 1, \frac{3}{2}$, and 2.
**Example 5**

**GRAPHING** \( y = a \cos bx \) **FOR** \( b \) **EQUAL TO A MULTIPLE OF** \( \pi \) (continued)

**Step 3**

Make a table of values determined by the \( x \)-values from Step 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{1}{2} )</th>
<th>1</th>
<th>( \frac{3}{2} )</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi x )</td>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( \cos \pi x )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(-3 \cos \pi x)</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
Example 5

Graphing $y = a \cos bx$ for $b$ equal to a multiple of $\pi$ (continued)

**Steps 4, 5**

Plot the points $(0, -3), \left(\frac{1}{2}, 0\right), (1, 3), \left(\frac{3}{2}, 0\right),$ and $(2, -3)$.

Join the points with a sinusoidal curve with amplitude $|\text{amplitude}| = 3$. The graph can be extended by repeating the cycle.
When $b$ is an integer multiple of $\pi$, the $x$-intercepts of the graph are rational numbers.
Example 6

INTERPRETING A SINE FUNCTION MODEL

The average temperature (in °F) at Mould Bay, Canada, can be approximated by the function

\[ f(x) = 34 \sin \left( \frac{\pi}{6} (x - 4.3) \right) \]

where \( x \) is the month and \( x = 1 \) corresponds to January, \( x = 2 \) corresponds to February, and so on.

(a) To observe the graph over a two-year interval and to see the maximum and minimum points, graph \( f \) in the window \([0, 25]\) by \([-45, 45]\).
The amplitude of the graph is 34 and the period is
\[
\frac{\frac{2\pi}{\pi}}{\frac{6}{6}} = 12.
\]
(b) What is the average temperature during the month of May?

May is month 5. Graph the function using a calculator, then find the value at $x = 5$.

Alternatively, use a calculator to compute $f(5)$.

$$f(5) = 34 \sin \left[ \frac{\pi}{6} (5 - 4.3) \right] \approx 12.18$$
(c) What would be an approximation for the average yearly temperature at Mould Bay?

From the graph, it appear that the average yearly temperature is about 0 °F since the graph is centered vertically about the line $y = 0$. 
4

Graphs of the Circular Functions
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Graphs of the Sine and Cosine Functions</td>
</tr>
<tr>
<td>4.2</td>
<td>Translations of the Graphs of the Sine and Cosine Functions</td>
</tr>
<tr>
<td>4.3</td>
<td>Graphs of the Tangent and Cotangent Functions</td>
</tr>
<tr>
<td>4.4</td>
<td>Graphs of the Secant and Cosecant Functions</td>
</tr>
<tr>
<td>4.5</td>
<td>Harmonic Motion</td>
</tr>
</tbody>
</table>
4.2 Translations of the Graphs of the Sine and Cosine Functions

- Horizontal Translations
- Vertical Translations
- Combinations of Translations
- Determining a Trigonometric Model Using Curve Fitting
Horizontal Translations

The graph of the function defined by \( y = f(x - d) \) is translated horizontally compared to the graph of \( y = f(x) \).

The translation is \( d \) units to the right if \( d > 0 \) and \(|d|\) units to the left if \( d < 0 \).
A horizontal translation is called a *phase shift*.

In the function $y = f(x - d)$, the expression $x - d$ is called the *argument*. 
Example 1

GRAPHING $y = \sin (x - d)$

Graph $y = \sin \left( x - \frac{\pi}{3} \right)$ over one period.

Method 1

To find an interval of one period, solve the three-part inequality $0 \leq x - \frac{\pi}{3} \leq 2\pi \Rightarrow \frac{\pi}{3} \leq x \leq \frac{7\pi}{3}$.

Divide this interval into four equal parts:

$\frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
GRAPHING $y = \sin(x - d)$ (continued)

Make a table of values determined by the $x$-values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\frac{4\pi}{3}$</th>
<th>$\frac{11\pi}{6}$</th>
<th>$\frac{7\pi}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x - \frac{\pi}{3}$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\sin\left(x - \frac{\pi}{3}\right)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>
Join the corresponding points with a smooth curve.

The period is $2\pi$, and the amplitude is 1.
**Method 2**

The argument $x - \frac{\pi}{3}$ indicates that the graph of $y = \sin x$ will be translated $\frac{\pi}{3}$ units to the right.
Example 2

**GRAPHING** \( y = a \cos (x - d) \)

Graph \( y = 3 \cos \left( x + \frac{\pi}{4} \right) \) over one period.

**Method 1**

To find an interval of one period, solve the three-part inequality \( 0 \leq x + \frac{\pi}{4} \leq 2\pi \Rightarrow -\frac{\pi}{4} \leq x \leq \frac{7\pi}{4} \).

Divide this interval into four equal parts:

\[
\frac{\pi}{3}, \quad \frac{5\pi}{6}, \quad \frac{4\pi}{3}, \quad \frac{11\pi}{6}, \quad \frac{7\pi}{3}
\]
Example 2

**GRAPHING** $y = a \cos (x - d)$ (continued)

Make a table of values determined by the $x$-values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\frac{\pi}{4}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{4}$</th>
<th>$\frac{7\pi}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + \frac{\pi}{4}$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\cos \left( x + \frac{\pi}{4} \right)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$3 \cos \left( x + \frac{\pi}{4} \right)$</td>
<td>3</td>
<td>0</td>
<td>$-3$</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
Example 2

**GRAPHING** \( y = a \cos (x - d) \) (continued)

Join the corresponding points with a smooth curve.

The period is \( 2\pi \), and the amplitude is 3.
**Example 2**

**GRAPHING** \( y = a \cos (x - d) \) (continued)

**Method 2**

\[
3 \cos \left( x + \frac{\pi}{4} \right) = 3 \cos \left[ x - \left( -\frac{\pi}{4} \right) \right]
\]

\( d = -\frac{\pi}{4} \), so the phase shift is \( \frac{\pi}{4} \) units to the left.
Graphing $y = a \cos b(x - d)$

Graph $y = -2\cos(3x + \pi)$ over two periods.

**Method 1**

To find an interval of one period, solve the three-part inequality $0 \leq 3x + \pi \leq 2\pi \Rightarrow -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

Divide this interval into four equal parts:

$-\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}$
Example 3

GRAPHING $y = a \cos b(x - d)$ (continued)

Make a table of values determined by the $x$-values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\frac{\pi}{3}$</th>
<th>$-\frac{\pi}{6}$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x + \pi$</td>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\cos(3x + \pi)$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$-2\cos(3x + \pi)$</td>
<td>$-2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$-2$</td>
</tr>
</tbody>
</table>
Join the corresponding points with a smooth curve. Then graph an additional half-period to the left and to the right.
Example 3

**GRAPHING** \( y = a \cos b(x - d) \)  (continued)

**Method 2**

Write the expression in the form \( a \cos b(x - d) \).

\[
y = -2 \cos(3x + \pi) = -2 \cos 3\left(x + \frac{\pi}{3}\right)
\]

Then \( a = -2 \), \( b = 3 \), and \( d = -\frac{\pi}{3} \).

The amplitude is \(|-2| = 2\), the period is \( \frac{2\pi}{3} \), and the phase shift is \( \frac{\pi}{3} \) units to the left as compared to the graph of \( y = -2 \cos 3x \).
Vertical Translations

The graph of the function defined by $y = c + f(x)$ is translated vertically compared to the graph of $y = f(x)$.

The translation is $c$ units up if $c > 0$ and $|c|$ units down if $c < 0$. 

Vertical translations of $y = f(x)$
Example 4

**GRAPHING** \( y = c + a \cos bx \)

Graph \( y = 3 - 2\cos 3x \) over one period.

The graph of \( y = 3 - 2\cos 3x \) is the same as the graph of \( y = -2\cos 3x \), vertically translated 3 units up.

The period of \( -2\cos 3x \) is \( \frac{2\pi}{3} \), so the key points have \( x \)-values

\[
0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}.
\]
GRAPHING $y = c + a \cos bx$ (continued)

Make a table of points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos 3x$</td>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$2\cos 3x$</td>
<td>2</td>
<td>0</td>
<td>−2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$3 - 2\cos 3x$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
Join the corresponding points with a smooth curve. Then graph an additional period to the left.
Guidelines for Sketching Graphs of Sine and Cosine Functions

**Method 1**

**Step 1** Find an interval whose length is one period \( \frac{2\pi}{b} \) by solving the three-part inequality \( 0 \leq b(x - d) \leq 2\pi \).

**Step 2** Divide the interval into four equal parts.

**Step 3** Evaluate the function for each of the five \( x \)-values resulting from Step 2. The points will be maximum points, minimum points, and points that intersect the line \( y = c \).
Step 4  Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5  Draw the graph over additional periods as needed.
Guidelines for Sketching Graphs of Sine and Cosine Functions

Method 2

First graph \( y = a \sin bx \) or \( y = a \cos bx \). The amplitude of the function is \(|a|\), and the period is \( \frac{2\pi}{b} \).

Use translations to graph the desired function. The vertical translation is \( c \) units up if \( c > 0 \) and \(|c|\) units down if \( c < 0 \).

The horizontal translation (phase shift) is \( d \) units to the right if \( d > 0 \) and \(|d|\) units to the left if \( d < 0 \).
Graphing $y = c + a \sin b(x - d)$

Graph $y = -1 + 2 \sin(4x + \pi)$ over two periods.

Use Method 1:

$$y = -1 + 2 \sin(4x + \pi) \implies y = 1 + 2 \sin \left[ 4 \left( x + \frac{\pi}{4} \right) \right]$$

**Step 1:** Find an interval whose length is one period.

$$0 \leq 4 \left( x + \frac{\pi}{4} \right) \leq 2\pi$$

$$0 \leq x + \frac{\pi}{4} \leq \frac{\pi}{2}$$

Divide each part by 4.

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$

Subtract $\frac{\pi}{4}$ from each part.
Example 5

GRAPHING \( y = c + a \sin b(x - d) \) (cont.)

**Step 2**: Divide the interval \( \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \) into four equal parts to get the \( x \)-values.

\[-\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{4}\]

**Step 3**: Make a table of values.
Example 5

**GRAPHING** \( y = c + a \sin b(x - d) \) (cont.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\frac{\pi}{4})</th>
<th>(-\frac{\pi}{8})</th>
<th>0</th>
<th>(\frac{\pi}{8})</th>
<th>(\frac{\pi}{4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + \frac{\pi}{4} )</td>
<td>0</td>
<td>(\frac{\pi}{8})</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{3\pi}{8})</td>
<td>(\frac{\pi}{2})</td>
</tr>
<tr>
<td>( 4\left(x + \frac{\pi}{4}\right) )</td>
<td>0</td>
<td>(\frac{\pi}{2})</td>
<td>(\pi)</td>
<td>(\frac{3\pi}{2})</td>
<td>(2\pi)</td>
</tr>
<tr>
<td>( \sin 4\left(x + \frac{\pi}{4}\right) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( 2\sin 4\left(x + \frac{\pi}{4}\right) )</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>(-1 + 2\sin(4x + \pi))</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
</tr>
</tbody>
</table>
Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. Extend the graph an additional half-period to the left and to the right to include two full periods.

\[ y = -1 + 2 \sin(4x + \pi) \]
The maximum average monthly temperature in New Orleans is 82°F and the minimum is 54°F. The table shows the average monthly temperatures.

<table>
<thead>
<tr>
<th>Month</th>
<th>°F</th>
<th>Month</th>
<th>°F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>54</td>
<td>July</td>
<td>82</td>
</tr>
<tr>
<td>Feb</td>
<td>55</td>
<td>Aug</td>
<td>81</td>
</tr>
<tr>
<td>Mar</td>
<td>61</td>
<td>Sept</td>
<td>77</td>
</tr>
<tr>
<td>Apr</td>
<td>69</td>
<td>Oct</td>
<td>71</td>
</tr>
<tr>
<td>May</td>
<td>73</td>
<td>Nov</td>
<td>59</td>
</tr>
<tr>
<td>June</td>
<td>79</td>
<td>Dec</td>
<td>55</td>
</tr>
</tbody>
</table>

The scatter diagram for a two-year interval suggests that the temperatures can be modeled with a sine curve.

(a) Using only the maximum and minimum temperatures, determine a function of the form

\[ f(x) = a \sin[b(x - d)] + c, \]

where \( a, b, c, \) and \( d \) are constants, that models the average monthly temperature in New Orleans. Let \( x \) represent the month, with January corresponding to \( x = 1 \).
Example 6

MODELING TEMPERATURE WITH A SINE FUNCTION (continued)

Use the maximum and minimum average monthly temperatures to determine the amplitude $a$:

$$a = \frac{82 - 54}{2} = 14$$

The average of the maximum and minimum temperatures gives $c$:

$$c = \frac{82 + 54}{2} = 68$$

Since temperatures repeat every 12 months, $b = \frac{2\pi}{12} = \frac{\pi}{6}$. 
To determine the phase shift, observe that the coldest month is January, when \( x = 1 \), and the hottest month is July, when \( x = 7 \). So choose \( d \) to be about 4.

The table shows that temperatures are actually a little warmer after July than before, so experiment with values just greater than 4 to find \( d \).

Trial and error using a calculator leads to \( d = 4.2 \).

\[
 f(x) = a \sin[b(x - d)] + c = 14 \sin\left[\frac{\pi}{6}(x - 4.2)\right] + 68
\]
(b) On the same coordinate axes, graph \( f \) for a two-year period together with the actual data values found in the table.

The graph shows the data points from the table, the graph of \( y = 14 \sin \left( \frac{\pi}{6} (x - 4.2) \right) + 68 \) and the graph of \( y = 14 \sin \frac{\pi}{6} x + 68 \) for comparison.
(c) Use the **sine regression** feature of a graphing calculator to determine a second model for these data.

\[
\begin{align*}
\text{SinReg} & \quad y = a \sin(bx + c) + d \\
a & = 14.39 \\
b & = 0.52 \\
c & = -2.15 \\
d & = 67.99
\end{align*}
\]

Values are rounded to the nearest hundredth.

\[
y = 14.39 \sin(0.52x - 2.15) + 67.99
\]
4

Graphs of the Circular Functions
4.1 Graphs of the Sine and Cosine Functions

4.2 Translations of the Graphs of the Sine and Cosine Functions

4.3 Graphs of the Tangent and Cotangent Functions

4.4 Graphs of the Secant and Cosecant Functions

4.5 Harmonic Motion
4.3 Graphs of the Tangent and Cotangent Functions

Graph of the Tangent Function
Graph of the Cotangent Function
Graphing Techniques
A **vertical asymptote** is a vertical line that the graph approaches but does not intersect, while function values increase or decrease without bound as $x$-values get closer and closer to the line.
# Tangent Function  \( f(x) = \tan x \)

**Domain:** \[ \left\{ x \mid x \neq (2n + 1) \frac{\pi}{2} \text{, where } n \text{ is any integer} \right\} \]

**Range:** \((\infty, \infty)\)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{\pi}{2})</td>
<td>undefined</td>
</tr>
<tr>
<td>(-\frac{\pi}{2})</td>
<td>(-1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{\pi}{4})</td>
<td>1</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td>undefined</td>
</tr>
</tbody>
</table>

\[ f(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \]
Tangent Function \( f(x) = \tan x \)

- The graph is discontinuous at values of \( x \) of the form \( x = (2n + 1) \frac{\pi}{2} \) and has vertical asymptotes at these values.
- Its \( x \)-intercepts are of the form \( x = n\pi \).
- Its period is \( \pi \).
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all \( x \) in the domain, \( \tan(-x) = -\tan(x) \).
Cotangent Function $f(x) = \cot x$

Domain: $\{x \mid x \neq n\pi, \text{where } n \text{ is any integer}\}$

Range: $(-\infty, \infty)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>undefined</td>
</tr>
</tbody>
</table>

$f(x) = \cot x$, $0 < x < \pi$
Cotangent Function \( f(x) = \cot x \)

- The graph is discontinuous at values of \( x \) of the form \( x = n\pi \) and has vertical asymptotes at these values.
- Its \( x \)-intercepts are of the form \( x = (2n + 1)\frac{\pi}{2} \).
- Its period is \( \pi \).
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all \( x \) in the domain, \( \cot(-x) = -\cot(x) \).
Tangent and Cotangent Functions

The tangent function can be graphed directly with a graphing calculator using the tangent key.

To graph the cotangent function, we must use one of the identities

$$\cot x = \frac{1}{\tan x} \quad \text{or} \quad \cot x = \frac{\cos x}{\sin x}$$

since graphing calculators generally do not have cotangent keys.
Guidelines for Sketching Graphs of Tangent and Cotangent Functions

**Step 1** Determine the period, $\frac{\pi}{b}$. To locate two adjacent vertical asymptotes, solve the following equations for $x$:

For $y = a \tan bx$: \( bx = -\frac{\pi}{2} \) and \( bx = \frac{\pi}{2} \)

For $y = a \cot bx$: \( bx = 0 \) and \( bx = \pi \)

**Step 2** Sketch the two vertical asymptotes found in Step 1.

**Step 3** Divide the interval formed by the vertical asymptotes into four equal parts.
Guidelines for Sketching Graphs of Tangent and Cotangent Functions

**Step 4** Evaluate the function for the first-quarter point, midpoint, and third-quarter point, using the $x$-values found in Step 3.

**Step 5** Join the points with a smooth curve, approaching the vertical asymptotes. Indicate additional asymptotes and periods of the graph as necessary.
Example 1

**GRAPHING** \( y = \tan bx \)

Graph \( y = \tan 2x \).

**Step 1** The period of this function is \( \frac{\pi}{2} \). To locate two adjacent vertical asymptotes, solve

\[
2x = -\frac{\pi}{2} \quad \text{and} \quad 2x = \frac{\pi}{2}
\]

The asymptotes have equations \( x = -\frac{\pi}{4} \) and \( x = \frac{\pi}{4} \).
Step 2  Sketch the two vertical asymptotes.
**Example 1**

**GRAPHING** \( y = \tan bx \) (continued)

**Step 3** Divide the interval \( \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \) into four equal parts.

first-quarter value: \(-\frac{\pi}{8}\)

middle value: 0

third-quarter value: \(\frac{\pi}{8}\)

**Step 4** Evaluate the function for the \( x \)-values found in Step 3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\frac{\pi}{8})</th>
<th>0</th>
<th>(\frac{\pi}{8})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x )</td>
<td>(-\frac{\pi}{4})</td>
<td>0</td>
<td>(\frac{\pi}{4})</td>
</tr>
<tr>
<td>tan (2x)</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
**Step 5** Join these points with a smooth curve, approaching the vertical asymptotes.

Draw another period by adding one-half period to the left and one-half period to the right.
Example 2

GRAPHING \( y = a \tan \, bx \)

Graph \( y = -3 \tan \frac{1}{2} x \).

The period is \( \frac{\pi}{2} \). To locate two adjacent vertical asymptotes, solve \( 2x = 0 \) and \( 2x = \square \) to obtain \( x = 0 \) and \( x = \frac{\pi}{2} \).

Divide the interval \( (0, \frac{\pi}{2}) \) into four equal parts to obtain the key \( x \)-values of \( \frac{\pi}{8}, \frac{\pi}{4}, \) and \( \frac{3\pi}{8} \).

Evaluate the function for the \( x \)-values found in Step 3 to obtain the key points

\[
\left( \frac{\pi}{8} , \frac{1}{2} \right), \left( \frac{\pi}{4} , 0 \right), \text{ and } \left( \frac{3\pi}{8} , -\frac{1}{2} \right).
\]
Example 2

GRAPHING $y = a \tan bx$ (continued)

Plot the asymptotes and the points found in step 4. Join them with a smooth curve.

Because the coefficient $-3$ is negative, the graph is reflected across the $x$-axis compared to the graph of $y = 3 \tan \frac{1}{2} x$. 
The function defined by $y = -3 \tan \frac{1}{2} x$ has a graph that compares to the graph of $y = \tan x$ as follows:

The period is larger because $b = \frac{1}{2}$, and $\frac{1}{2} < 1$.

The graph is “stretched” because $a = -3$, and $|\ -3\ | > 1$. 
Each branch of the graph goes down from left to right (the function decreases) between each pair of adjacent asymptotes because \( a = -3 \), and \(-3 < 0\).

When \( a < 0 \), the graph is reflected across the \( x \)-axis compared to the graph of \( y = |a| \tan bx \).
Example 3

GRAPHING \( y = a \cot bx \)

Graph \( y = \frac{1}{2} \cot 2x \).

The period is \( \frac{\pi}{1/2} = 2\pi \). Adjacent vertical asymptotes are at \( x = -\pi \) and \( x = -\pi/2 \).

Divide the interval \((-\pi, \pi)\) into four equal parts to obtain the key \( x \)-values of \(-\frac{\pi}{2}, 0, \) and \( \frac{\pi}{2} \).

Evaluate the function for the \( x \)-values found in Step 3 to obtain the key points

\[ \left(-\frac{\pi}{2}, 3\right), (0,0), \text{ and } \left(\frac{\pi}{2}, -3\right). \]
Example 3

GRAPHING $y = a \cot bx$ (continued)

Plot the asymptotes and the points found in step 4. Join them with a smooth curve.
Graph \( y = 2 + \tan x \).

Every \( y \) value for this function will be 2 units more than the corresponding \( y \) value in \( y = \tan x \), causing the graph to be translated 2 units up compared to \( y = \tan x \).
To see the vertical translation, observe the coordinates displayed at the bottoms of the screens.
Example 5

GRAPHING A COTANGENT FUNCTION WITH VERTICAL AND HORIZONTAL TRANSLATIONS

Graph \( y = -2 - \cot\left(x - \frac{\pi}{4}\right) \).

The period is \( \pi \) because \( b = 1 \).

The graph will be translated down two units because \( c = -2 \).

The graph will be reflected across the \( x \)-axis because \( a = -1 \).

The phase shift is \( \frac{\pi}{4} \) units to the right.
Example 5

GRAPHING A COTANGENT FUNCTION WITH VERTICAL AND HORIZONTAL TRANSLATIONS (continued)

To locate adjacent asymptotes, solve

\[ x - \frac{\pi}{4} = 0 \Rightarrow x = \frac{\pi}{4} \quad \text{and} \quad x - \frac{\pi}{4} = \pi \Rightarrow x = \frac{5\pi}{4}. \]

Divide the interval \( \left( \frac{\pi}{4}, \frac{5\pi}{4} \right) \) into four equal parts to obtain the key \( x \)-values

\[ \frac{\pi}{2}, \frac{3\pi}{4}, \text{ and } \pi. \]

Evaluate the function for the key \( x \)-values to obtain the key points

\( \left( \frac{\pi}{2}, -3 \right), \left( \frac{3\pi}{4}, -2 \right), \text{ and } (\pi, -1). \)
Plot the asymptotes and key points, then join them with a smooth curve.

An additional period to the left has been graphed.
4

Graphs of the Circular Functions
4 Graphs of the Circular Functions

4.1 Graphs of the Sine and Cosine Functions

4.2 Translations of the Graphs of the Sine and Cosine Functions

4.3 Graphs of the Tangent and Cotangent Functions

4.4 Graphs of the Secant and Cosecant Functions

4.5 Harmonic Motion
4.4 Graphs of the Secant and Cosecant Functions

- Graph of the Secant Function
- Graph of the Cosecant Function
- Graphing Techniques
- Addition of Ordinates
- Connecting Graphs with Equations
Secant Function  \( f(x) = \sec x \)

Domain: \( \left\{ x \mid x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer} \right\} \)

Range: \( (-\infty, -1] \cup [1, \infty) \)

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>(-\frac{\pi}{2})</td>
<td>undefined</td>
<td>(\frac{\pi}{2})</td>
<td>undefined</td>
</tr>
<tr>
<td>(-\frac{\pi}{4})</td>
<td>(\sqrt{2})</td>
<td>(\frac{3\pi}{4})</td>
<td>(-\sqrt{2})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>(\frac{\pi}{2})</td>
<td>(-1)</td>
</tr>
<tr>
<td>(\frac{\pi}{4})</td>
<td>(\sqrt{2})</td>
<td>(\frac{3\pi}{2})</td>
<td>undefined</td>
</tr>
</tbody>
</table>
Secant Function \( f(x) = \sec x \)
Secant Function \( f(x) = \sec x \)

- The graph is discontinuous at values of \( x \) of the form \( x = (2n + 1) \frac{\pi}{2} \) and has vertical asymptotes at these values.
- There are no \( x \)-intercepts.
- Its period is \( 2\pi \).
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the \( y \)-axis, so the function is an even function. For all \( x \) in the domain, \( \sec(-x) = \sec(x) \).
Cosecant Function  \( f(x) = \csc x \)

**Domain:**  \( \{ x \mid x \neq n\pi, \text{ where } n \text{ is any integer} \} \)

**Range:**  \( (-\infty, \infty) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>undefined</td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>2</td>
<td>( \pi )</td>
<td>undefined</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>( 2\pi )</td>
<td>undefined</td>
</tr>
</tbody>
</table>
Cosecant Function \( f(x) = \csc x \)
Cotangent Function $f(x) = \cot x$

- The graph is discontinuous at values of $x$ of the form $x = n\pi$ and has vertical asymptotes at these values.
- There are no $x$-intercepts.
- Its period is $2\pi$.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all $x$ in the domain, $\csc(-x) = -\csc(x)$. 
Secant and Cosecant Functions

To graph the cosecant function on a graphing calculator, use the identity

\[ \csc x = \frac{1}{\sin x} \]

To graph the secant function on a graphing calculator, use the identity

\[ \sec x = \frac{1}{\cos x} \]
Secant and Cosecant Functions

$Y_1 = \sin X$  $Y_2 = \csc X$

Y_1 = \cos X  Y_2 = \sec X

Trig window; connected mode

Trig window; connected mode
Step 1 Graph the corresponding reciprocal function as a guide, use a dashed curve.

<table>
<thead>
<tr>
<th>To Graph</th>
<th>Use as a Guide</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = a \csc , bx$</td>
<td>$y = a \sin , bx$</td>
</tr>
<tr>
<td>$y = a \sec , bx$</td>
<td>$y = \cos , bx$</td>
</tr>
</tbody>
</table>

Step 2 Sketch the vertical asymptotes. They will have equations of the form $x = k$, where $k$ is an $x$-intercept of the graph of the guide function.
Step 3  Sketch the graph of the desired function by drawing the typical U-shaped branches between the adjacent asymptotes.

The branches will be above the graph of the guide function when the guide function values are positive and below the graph of the guide function when the guide function values are negative.
Example 1

GRAPHING $y = a \sec bx$

Graph $y = 2 \sec \frac{1}{2} x$.

**Step 1** Graph the corresponding reciprocal function $y = 2 \cos \frac{1}{2} x$.

The function has amplitude 2 and one period of the graph lies along the interval that satisfies the inequality

$$0 \leq \frac{1}{2} x \leq 2\pi \quad \text{or} \quad [0, 4\pi]$$

Divide the interval into four equal parts and determine the key points.

$$(0, 2), \ (\pi, 0), \ (2\pi, -2), \ (3\pi, 0), \ (4\pi, 2)$$
Step 2 Sketch the vertical asymptotes. These occur at $x$-values for which the guide function equals 0, such as $x = -3\pi, x = 3\pi, x = \pi, x = 3\pi$. 
Step 3 Sketch the graph of $y = 2 \sec \frac{1}{2}x$ by drawing the typical U-shaped branches, approaching the asymptotes.
Example 2

GRAPHING $y = a \csc(x - d)$

Graph $y = \frac{3}{2} \csc \left( x - \frac{\pi}{2} \right)$.

**Step 1** Graph the corresponding reciprocal function

$y = \frac{3}{2} \sin \left( x - \frac{\pi}{2} \right)$. 

![Graph of $y = \frac{3}{2} \sin \left( x - \frac{\pi}{2} \right)$]

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Step 2 Sketch the vertical asymptotes through the $x$-intercepts of $y = \frac{3}{2} \sin\left(x - \frac{\pi}{2}\right)$. These have the form $x = \frac{\pi}{2}(2n + 1)$ where $n$ is any integer.
Example 2

GRAPHING \( y = a \csc (x - d) \) (continued)

**Step 3** Sketch the graph of \( y = \frac{3}{2} \csc \left( x - \frac{\pi}{2} \right) \) by drawing typical U-shaped branches between the asymptotes.
Addition of Ordinates

New functions can be formed by adding or subtracting other functions.

\[ y = \cos x + \sin x \]

Graph \( y = \cos x \) and \( y = \sin x \). Then, for selected values of \( x \), plot the points \((x, \cos x + \sin x)\) and join the points with a sinusoidal curve.
Example 3(a) DETERMINING AN EQUATION FOR A GRAPH

Determine an equation for the graph.

This is the graph of $y = \tan x$ reflected across the $y$-axis and stretched vertically by a factor of 2.

An equation for the graph is $y = -2 \tan x$. 
Example 3(b) DETERMINING AN EQUATION FOR A GRAPH

Determine an equation for the graph.

This is the graph of \( y = \cot x \) with period \( \frac{\pi}{2} \). Therefore, \( b = 2 \).

An equation for the graph is \( y = \cot 2x \).
Determine an equation for the graph.

This is the graph of \( y = \sec x \), translated one unit upward.

An equation for the graph is \( y = 1 + \sec x \).
4

Graphs of the Circular Functions
4 Graphs of the Circular Functions

4.1 Graphs of the Sine and Cosine Functions

4.2 Translations of the Graphs of the Sine and Cosine Functions

4.3 Graphs of the Tangent and Cotangent Functions

4.4 Graphs of the Secant and Cosecant Functions

4.5 Harmonic Motion
4.5 Harmonic Motion

Simple Harmonic Motion    Damped Oscillatory Motion
Simple Harmonic Motion

The position of a point oscillating about an equilibrium position at time $t$ is called **simple harmonic motion**.
Simple Harmonic Motion

The position of a point oscillating about an equilibrium position at time $t$ is modeled by either

$$s(t) = a \cos \omega t \quad \text{or} \quad s(t) = a \sin \omega t$$

where $a$ and $\omega$ are constants, with $\omega > 0$.

The amplitude of the motion is $|a|$, the period is $\frac{2\pi}{\omega}$, and the frequency is $\frac{\omega}{2\pi}$ oscillations per time unit.
Suppose that an object is attached to a coiled spring. It is pulled down a distance of 5 in. from its equilibrium position and then released. The time for one complete oscillation is 4 seconds.

(a) Give an equation that models the position of the object at time $t$.

When the object is released at $t = 0$, the object is at distance $-4$ in. from equilibrium.

$$s(0) = -4 = a \cos \omega(0) \Rightarrow a = -1$$
Since the time needed to complete one oscillation is 4 sec, $P = 4$, so

$$4 = \frac{2\pi}{\omega} \implies \omega = \frac{2\pi}{4} = \frac{\pi}{2}.$$ 

The motion is modeled by $\sin(t) = -5\cos\frac{\pi}{2}t$. 
Example 1

MODELING THE MOTION OF A SPRING
(continued)

(b) Determine the position at $t = 1.5$ sec.

$$\sin(1.5) = -5\cos\left(\frac{\pi}{2}(1.5)\right) \approx 3.54 \text{ in.}$$

At $t = 1.5$ seconds, the object is about 3.54 in. above the equilibrium position.

(c) Find the frequency.

The frequency is the reciprocal of the period.

$$\text{Frequency} = \frac{1}{4} \text{ oscillation per second}$$
Example 2

ANALYZING HARMONIC MOTION

Suppose that an object oscillates according to the model \( s(t) = 8 \sin 3t \), where \( t \) is in seconds and \( s(t) \) is in feet. Analyze the motion.

The motion is harmonic because the model is of the form \( y = a \sin \omega t \).

\[ a = 8, \text{ so the object oscillates 8 feet in either direction from the starting point}. \]

\[ \text{Period } = \frac{2\pi}{3} \approx 2.1 \text{ sec} = \text{time (in seconds) for one complete oscillation} \]

\[ \text{Frequency } = \frac{3}{2\pi} \approx .48 \text{ oscillation per sec} \]
Most oscillatory motions are \textit{damped} by the effect of friction which causes the amplitude of the motion to diminish gradually until the weight comes to rest.

This motion can be modeled by

\[ s(t) = e^{-t} \sin t \]
Damped Oscillatory Motion

\[ y_3 = e^{-x} \sin x \]
\[ y_1 = e^{-x} \]
\[ y_2 = -e^{-x} \]

\[ y = e^{-t} \]
\[ s(t) = e^{-t} \sin t \]
\[ y = \sin t \]
\[ y = -e^{-t} \]

\[ t \]
\[ y \]
\[ x \]
\[ 2\pi \]
\[ 0 \]
\[ -0.2 \]
\[ 0.5 \]