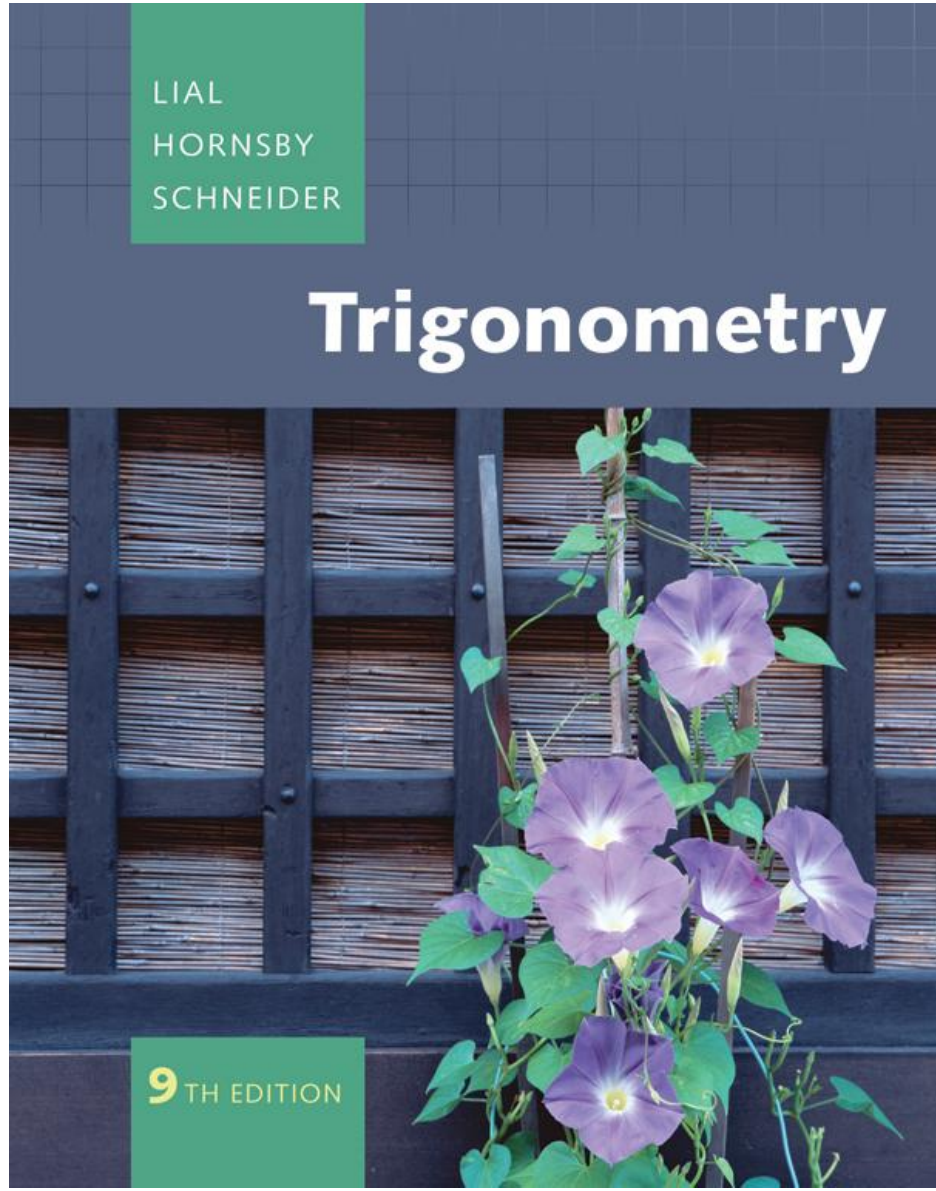
8

Complex Numbers, Polar Equations, and Parametric Equations



8 Complex Numbers, Polar Equations, and Parametric Equations

8.1 Complex Numbers

Basic Concepts of Complex Numbers ▪ Complex Solutions of Equations ▪ Operations on Complex Numbers

Basic Concepts of Complex Numbers

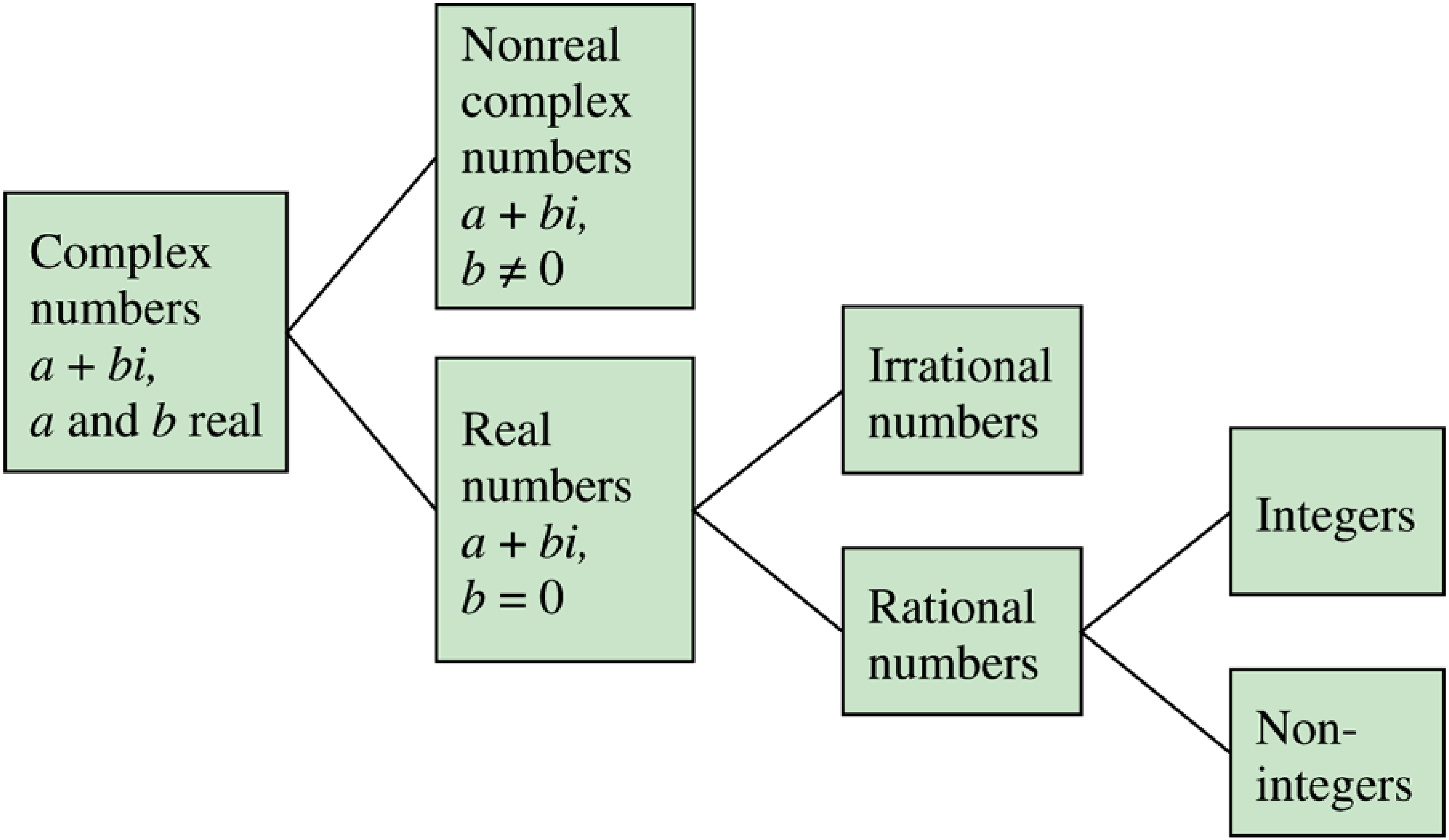
$$i^2 = -1 \Rightarrow i = \sqrt{-1}$$

- i is called the **imaginary unit**.
- Numbers of the form $a + bi$ are called **complex numbers**.

  a is the **real part**.

  b is the **imaginary part**.

- $a + bi = c + di$ **if and only if** $a = c$ **and** $b = d$.


Complex numbers
$a + bi$,
a and b real
Nonreal complex numbers
$a + bi$,
$b \neq 0$
Real numbers
$a + bi$,
$b = 0$
Irrational numbers
Rational numbers
Integers
Non-integers

Basic Concepts of Complex Numbers

- If $a = 0$ and $b \neq 0$, the complex number is a **pure imaginary number**.

  Example: $3i$

- If $a \neq 0$ and $b \neq 0$, the complex number is a **nonreal complex number**.

  Example: $7 + 2i$

- A complex number written in the form $a + bi$ or $a + ib$ is written in **standard form**.

The Expression $\sqrt{-a}$

If $a > 0$, then $\sqrt{-a} = i\sqrt{a}$.

Example 1 WRITING $\sqrt{-a}$ AS $i\sqrt{a}$

Write as the product of real number and *i*, using the definition of $\sqrt{-a}$.

(a) $\sqrt{-16} = i\sqrt{16} = 4i$

(b) $\sqrt{-70} = i\sqrt{70}$

(c) $\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}$

Example 2 SOLVING QUADRATIC EQUATIONS FOR COMPLEX SOLUTIONS

Solve each equation.

(a) $x^2 = -9$

$$x = \pm\sqrt{-9}$$

$$x = \pm i\sqrt{9}$$

$$x = \pm 3i$$

Solution set: $\{\pm 3i\}$

(b) $x^2 + 24 = 0$

$$x^2 = -24$$

$$x = \pm\sqrt{-24}$$

$$x = \pm i\sqrt{24}$$

$$x = \pm i\sqrt{4 \cdot 6}$$

$$x = \pm 2i\sqrt{6}$$

Solution set: $\{\pm 2i\sqrt{6}\}$

Example 3 **SOLVING QUADRATIC EQUATIONS FOR COMPLEX SOLUTIONS**

Solve $9x^2 + 5 = 6x.$

Write the equation in standard form, $9x^2 - 6x + 5 = 0$ then solve using the quadratic formula with $a = 9$, $b = -6$, and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)}$$

$$= \frac{6 \pm \sqrt{-144}}{18} = \frac{6 \pm 12i}{18} = \frac{1 \pm 2i}{3} = \frac{1}{3} \pm \frac{2i}{3}$$

Solution set: $\left\{\frac{1}{3} \pm \frac{2}{3} i\right\}$

Caution

When working with negative radicands, use the definition $\sqrt{-a} = i\sqrt{a}$ before using any of the other rules for radicals.

In particular, the rule $\sqrt{c} \cdot \sqrt{d} = \sqrt{cd}$ is valid only when *c* and *d* are **not** both negative.

For example $\sqrt{(-4)(-9)} = \sqrt{36} = 6$,
while $\sqrt{-4} \cdot \sqrt{-9} = 2i(3i) = 6i^2 = -6$,
so $\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{(-4)(-9)}$.

▶ Example 4

FINDING PRODUCTS AND QUOTIENTS INVOLVING NEGATIVE RADICANDS

Multiply or divide as indicated. Simplify each answer.

(a) $\sqrt{-7} \cdot \sqrt{-7} = i\sqrt{7} \cdot i\sqrt{7} = 7i^2 = -7$

(b) $\sqrt{-6} \cdot \sqrt{-10} = i\sqrt{6} \cdot i\sqrt{10} = i^2\sqrt{60} = i^2\sqrt{4 \cdot 15}$

$= 2i^2\sqrt{15} = -2\sqrt{15}$

(c) $\dfrac{\sqrt{-20}}{\sqrt{-2}} = \dfrac{i\sqrt{20}}{i\sqrt{2}} = \sqrt{10}$

(d) $\dfrac{\sqrt{-48}}{\sqrt{24}} = \dfrac{i\sqrt{48}}{\sqrt{24}} = i\sqrt{2}$

Example 5 SIMPLIFYING A QUOTIENT INVOLVING A NEGATIVE RADICAND

Write $\dfrac{-8+\sqrt{-128}}{4}$ in standard form *a* + *bi*.

$$\frac{-8+\sqrt{-128}}{4} = \frac{-8+\sqrt{-64\cdot 2}}{4}$$

$$= \frac{-8+8i\sqrt{2}}{4} \qquad \sqrt{-64} = 8i$$

$$= \frac{4\left(-2+2i\sqrt{2}\right)}{4} \qquad \text{Factor.}$$

$$= -2+2i\sqrt{2}$$

Addition and Subtraction of Complex Numbers

For complex numbers *a* + *bi* and *c* + *di*,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Example 6

ADDING AND SUBTRACTING COMPLEX NUMBERS

Find each sum or difference.

(a) $(3 - 4i) + (-2 + 6i) = [3 + (-2)] + (-4i + 6i)$

$= 1 + 2i$

(b) $(-9 + 7i) + (3 - 15i) = -6 - 8i$

(c) $(-4 + 3i) - (6 - 7i) = (-4 - 6) + [3 - (-7)]i$

$= -10 + 10i$

(d) $(12 - 5i) - (8 - 3i) + (-4 + 2i)$

$= (12 - 8 - 4) + (-5 + 3 + 2)i$

$= 0 + 0i = 0$

Multiplication of Complex Numbers

$$
\begin{aligned}
(a+bi)(c+di) &= ac + adi + bci + bdi^2 \\
&= ac + (ad+bc)i + bd(-1) \\
&= (ac-bd) + (ad+bc)i
\end{aligned}
$$

Example 7 MULTIPLYING COMPLEX NUMBERS

Find each product.

(a) $(2 - 3i)(3 + 4i) = 2(3) + (2)(4i) + (-3i)(3) + (-3i)(4i)$

$= 6 + 8i - 9i - 12i^2$

$= 6 - i - 12(-1)$

$= 18 - i$

(b) $(4 + 3i)^2 = 4^2 + 2(4)(3i) + (3i)^2$

$= 16 + 24i + 9i^2$

$= 16 + 24i + 9(-1)$

$= 7 + 24i$

Example 7

MULTIPLYING COMPLEX NUMBERS

(c) $(2 + i)(-2 - i)$

$= -4 - 2i - 2i - i^2$

$= -4 - 4i - (-1)$

$= -4 - 4i + 1$

$= -3 - 4i$

(d) $(6 + 5i)(6 - 5i)$

$= 6^2 - (5i)^2$

$= 36 - 25i^2$

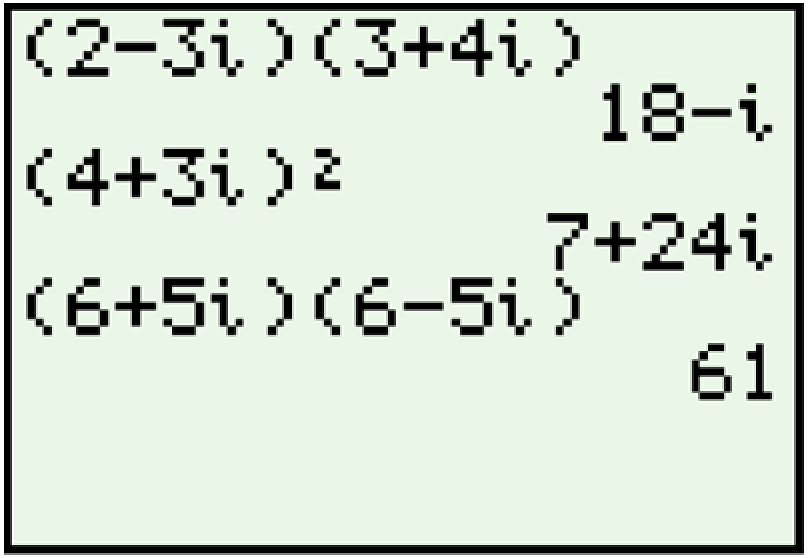
$= 36 - 25(-1)$

$= 36 + 25$

$= 61$ or $61 + 0i$

▶ Example 7 MULTIPLYING COMPLEX NUMBERS (cont.)

This screen shows how the TI–83/84 Plus displays the results found in parts (a), (b), and (d) in this example.



Example 8 SIMPLIFYING POWERS OF *i*

Simplify each power of *i*.

(a) i^{15} (b) i^{-3} (c) $\dfrac{1}{i^{-13}}$

Write the given power as a product involving $i^2 = -1$ or $i^4 = 1$.

(a) $i^{15} = i^{12} \cdot i^3 = \left(i^4\right)^3 \left(i^2 \cdot i\right) = 1^3(-i) = -i$

(b) $i^{-3} = i^{-4} \cdot i = \left(i^4\right)^{-1} \cdot i = 1^{-1} \cdot i = i$

(c) $\dfrac{1}{i^{-13}} = i^{13} = i^{12} \cdot i = \left(i^4\right)^3 \cdot i = 1^3 \cdot i = i$

Powers of i

$i^1 = i$	$i^5 = i$	$i^9 = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$, and so on.

Property of Complex Conjugates

For real numbers *a* and *b*,

$$(a+bi)(a-bi)=a^2+b^2$$

Example 9(a) DIVIDING COMPLEX NUMBERS

Write the quotient in standard form *a* + *bi*.

$$\frac{3+2i}{5-i} = \frac{3+2i}{5-i}\cdot\frac{5+i}{5+i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$= \frac{15+3i+10i+2i^2}{25-i^2}$$

Multiply.

$$= \frac{13+13i}{26}$$

$i^2 = -1$

$$= \frac{13(1+1i)}{26}$$

Factor.

$$= \frac{1}{2}+\frac{1}{2}i$$

Lowest terms; standard form

Example 9(b) DIVIDING COMPLEX NUMBERS

Write the quotient in standard form *a* + *bi*.

$$\frac{3}{i} = \frac{3}{i} \cdot \frac{-i}{-i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

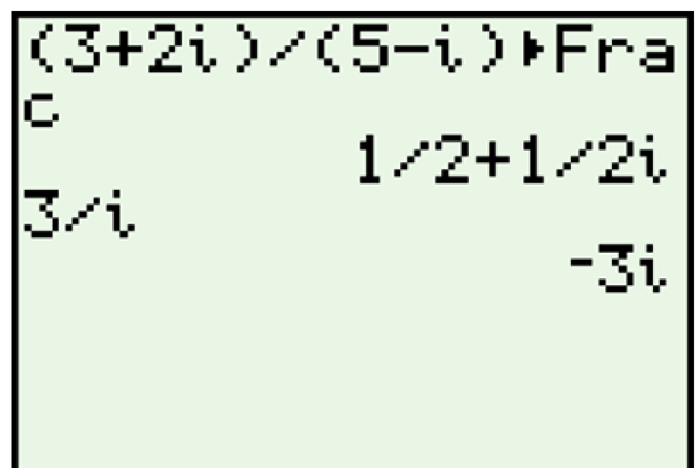
$$= \frac{-3i}{-i^2} = \frac{-3i}{1}$$

Multiply. $i^2 = -1$

$$= -3i \text{ or } 0 - 3i$$

Standard form

This screen shows how the TI–83/84 Plus displays the results in this example.



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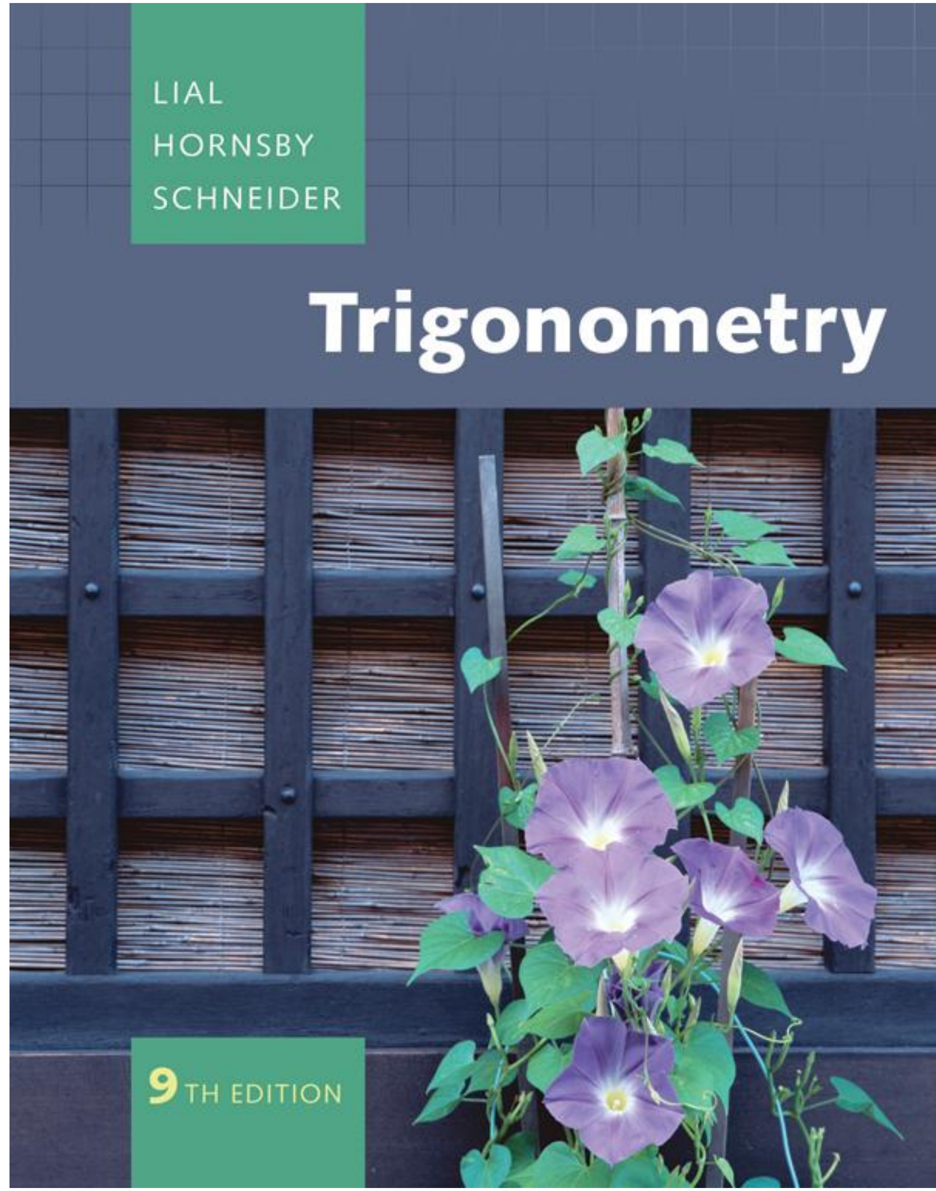
Complex Numbers, Polar Equations, and Parametric Equations



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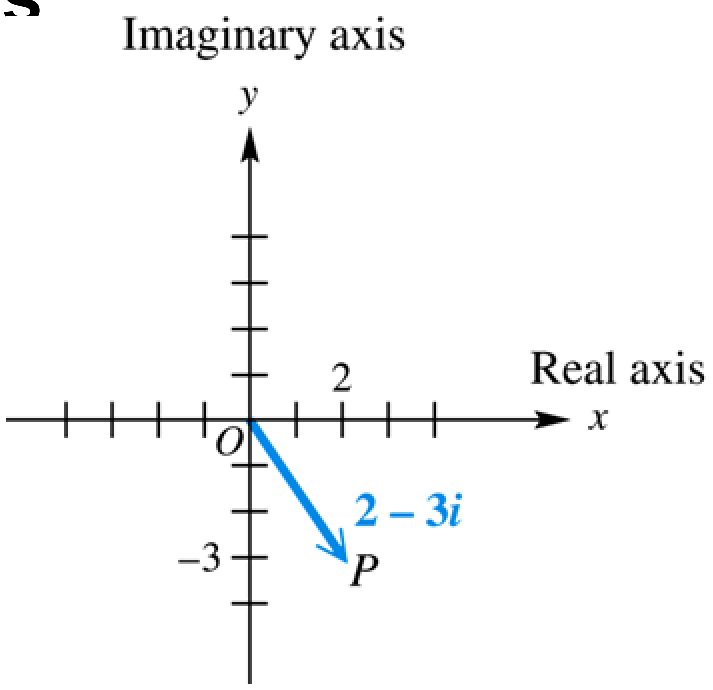
8 Complex Numbers, Polar Equations, and Parametric Equations

8.2 Trigonometric (Polar) Form of Complex Numbers

The Complex Plane and Vector Representation ▪ Trigonometric (Polar) Form ▪ Converting Between Rectangular and Trigonometric (Polar) Forms ▪ An Application of Complex Numbers to Fractals

The Complex Plane and Vector Representation

- Horizontal axis: **real axis**
- Vertical axis: **imaginary axis**

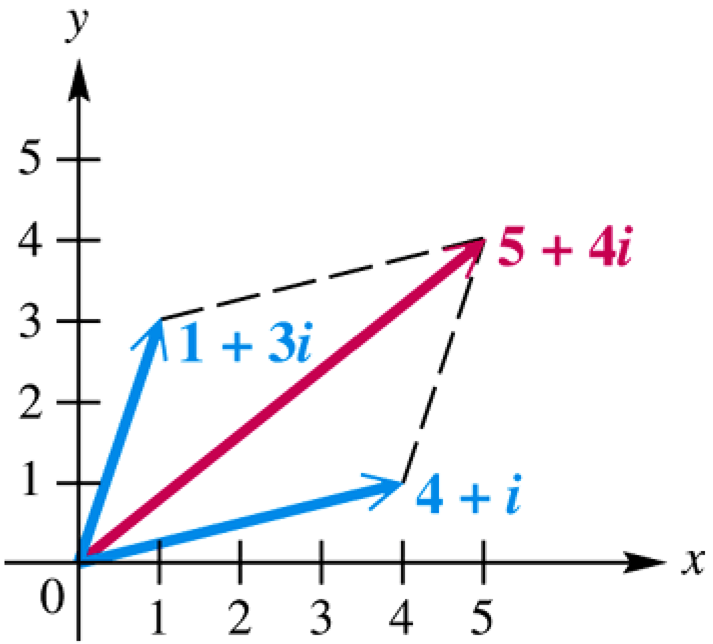


Each complex number $a + bi$ determines a unique position vector with initial point (0, 0) and terminal point (a, b).

The Complex Plane and Vector Representation

The sum of two complex numbers is represented by the vector that is the resultant of the vectors corresponding to the two numbers.

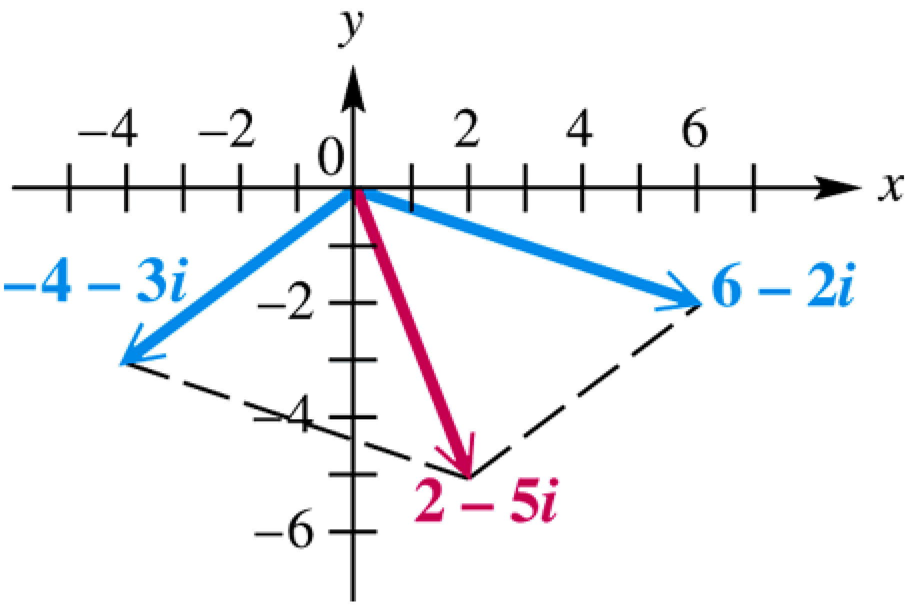
$(4 + i) + (1 + 3i) = 5 + 4i$



▶ Example 1 EXPRESSING THE SUM OF COMPLEX NUMBERS GRAPHICALLY

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

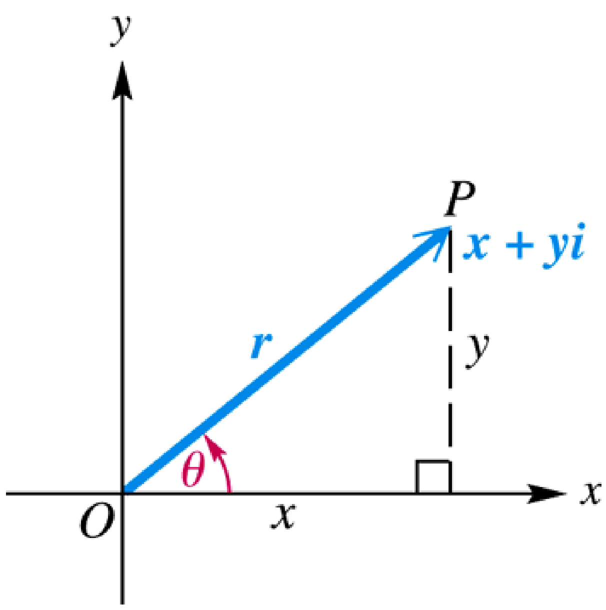
$$(6 - 2i) + (-4 - 3i) = 2 - 5i$$



Relationships Among x, y, r, and θ.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}, \text{ if } x \neq 0$$



Trigonometric (Polar) Form of a Complex Number

The expression $\boldsymbol{r(\cos\theta + i\sin\theta)}$ is called the **trigonometric form** (or **polar form**) of the complex number $x + yi$.

The expression $\boldsymbol{\cos\theta + i\sin\theta}$ is sometimes abbreviated **cis** $\boldsymbol{\theta}$.

Using this notation, $\boldsymbol{r(\cos\theta + i\sin\theta)}$ is written $\boldsymbol{r}$ **cis** $\boldsymbol{\theta}$.

The number r is the **absolute value** (or **modulus**) of $x + yi$, and θ is the **argument** of $x + yi$.

Example 2 CONVERTING FROM TRIGONOMETRIC FORM TO RECTANGULAR FORM

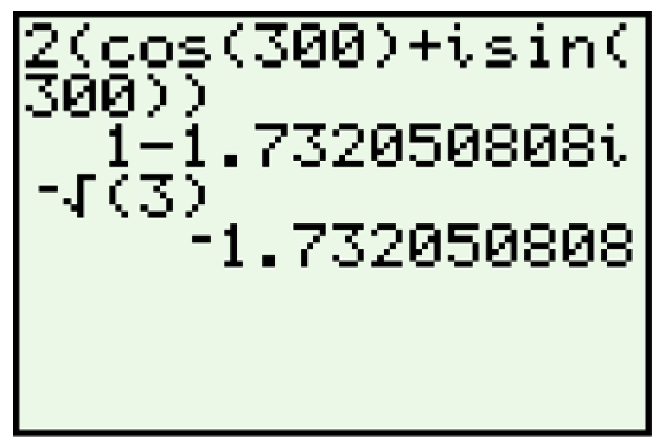
Express 2(cos 300° + *i* sin 300°) in rectangular form.

$$\cos 300° = \frac{1}{2} \qquad \sin 300° = -\frac{\sqrt{3}}{2}$$

$$2(\cos 300° + i\sin 300°) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 1 - i\sqrt{3}$$

The graphing calculator screen confirms the algebraic solution. The imaginary part is an approximation for $-\sqrt{3}$.



Converting From Rectangular Form to Trigonometric Form

Step 1 Sketch a graph of the number $x + yi$ in the complex plane.

Step 2 Find r by using the equation

$$r = \sqrt{x^2 + y^2}.$$

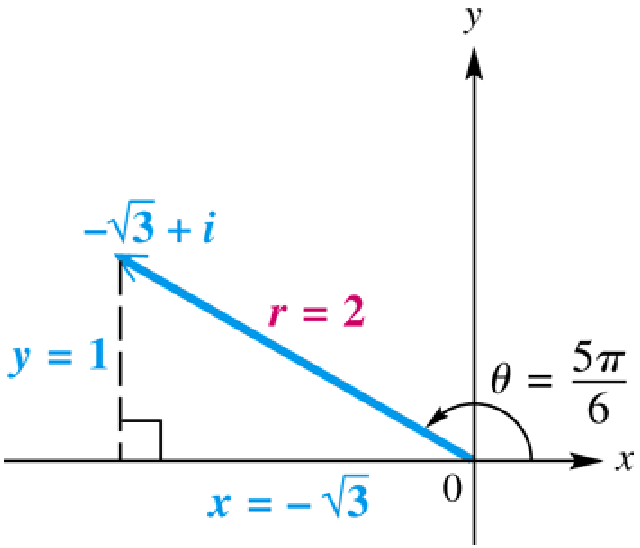
Step 3 Find θ by using the equation $\tan\theta = \frac{y}{x}, x \neq 0$, choosing the quadrant indicated in Step 1.

▶ Caution

Be sure to choose the correct quadrant for θ by referring to the graph sketched in Step 1.

▶ Example 3(a) CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure.)



***Step 1*:**

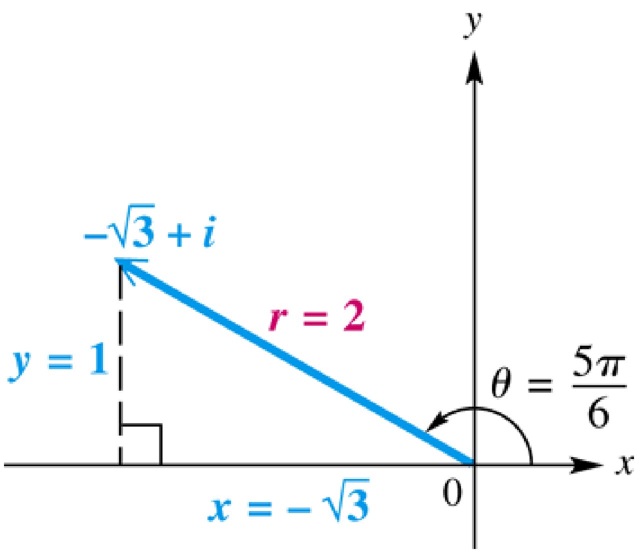
Sketch the graph of $-\sqrt{3}+i$ in the complex plane.

***Step 2*:**

$$x=-\sqrt{3} \text{ and } y=1, \text{ so } r=\sqrt{x^2+y^2}=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$$

CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM (continued)

***Step 3*:** $\tan\theta = \dfrac{y}{x} = \dfrac{1}{-\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$



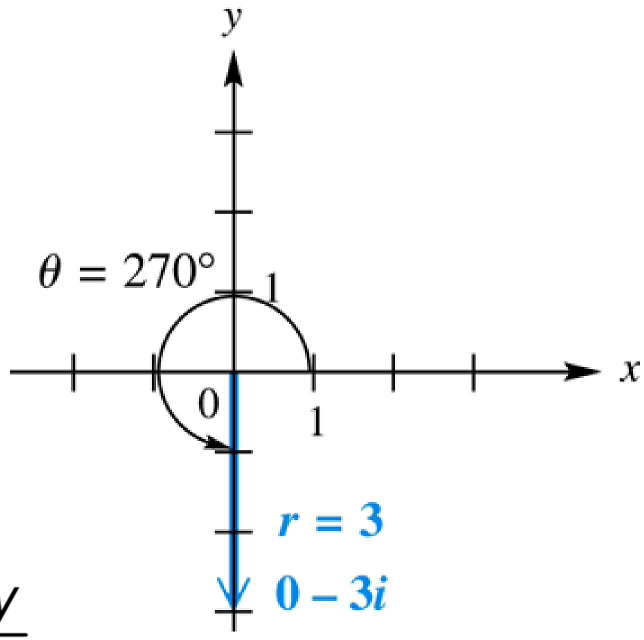
The reference angle for θ is $\dfrac{\pi}{6}$.

The graph shows that θ is in quadrant II, so $\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$.

▶ Example 3(b) CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM

Write $-3i$ in trigonometric form. (Use degree measure.)

Sketch the graph of $-3i$ in the complex plane.



$x = 0$ and $y = -3$, so

$$r = \sqrt{0^2 + (-3)^2} = 3.$$

We cannot find θ by using $\tan\theta = \frac{y}{x}$, because $x = 0$.

From the graph, a value for θ is 270 .

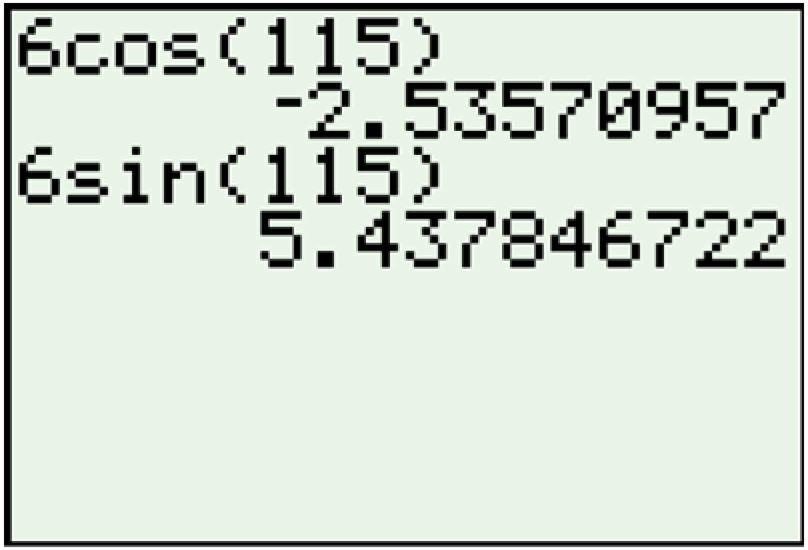
$$-3 = 3(\cos 270° + i \sin 270°) = 3 \text{ cis } 270°$$

Example 4

CONVERTING BETWEEN TRIGONOMETRIC AND RECTANGULAR FORMS USING CALCULATOR APPROXIMATIONS

Write each complex number in its alternative form, using calculator approximations as necessary.

(a) $6(\cos 115\ \ + i \sin 115\ \ ) \approx -2.5357 + 5.4378i$

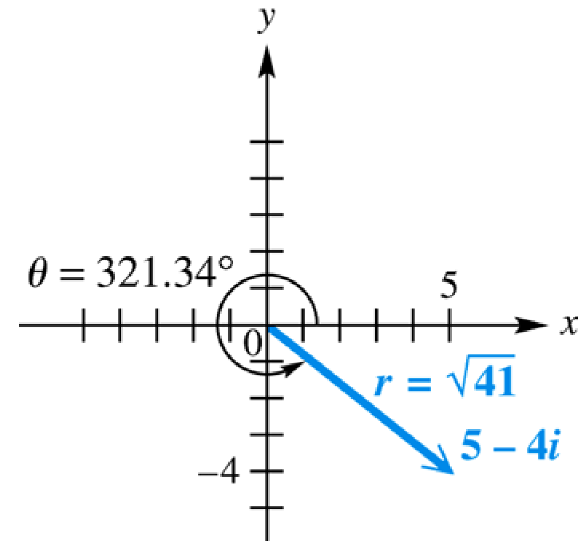


Example 4

CONVERTING BETWEEN TRIGONOMETRIC AND RECTANGULAR FORMS USING CALCULATOR APPROXIMATIONS (continued)

(b) $5 - 4i$

A sketch of $5 - 4i$ shows that θ must be in quadrant IV.



$$r = \sqrt{5^2 + (-4)^2} = \sqrt{41}$$

$$\tan\theta = -\frac{4}{5}$$

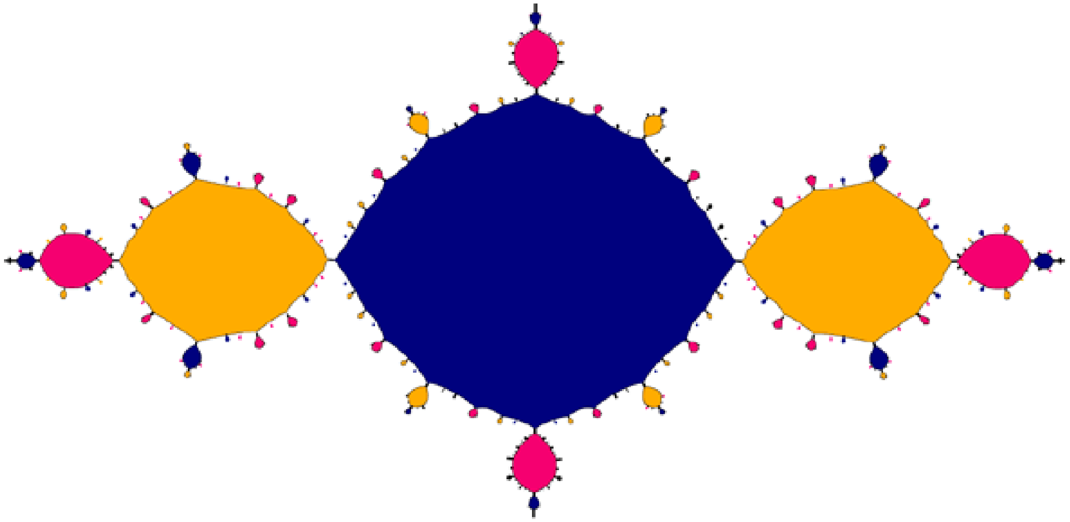
The reference angle for θ is approximately –38.66 . The graph shows that θ is in quadrant IV, so θ = 360 – 38.66 = 321.34 .

$$5 - 4i = \sqrt{41}\,\text{cis}\,321.34°$$

Example 5

DECIDING WHETHER A COMPLEX NUMBER IS IN THE JULIA SET

The figure shows the fractal called the Julia set.



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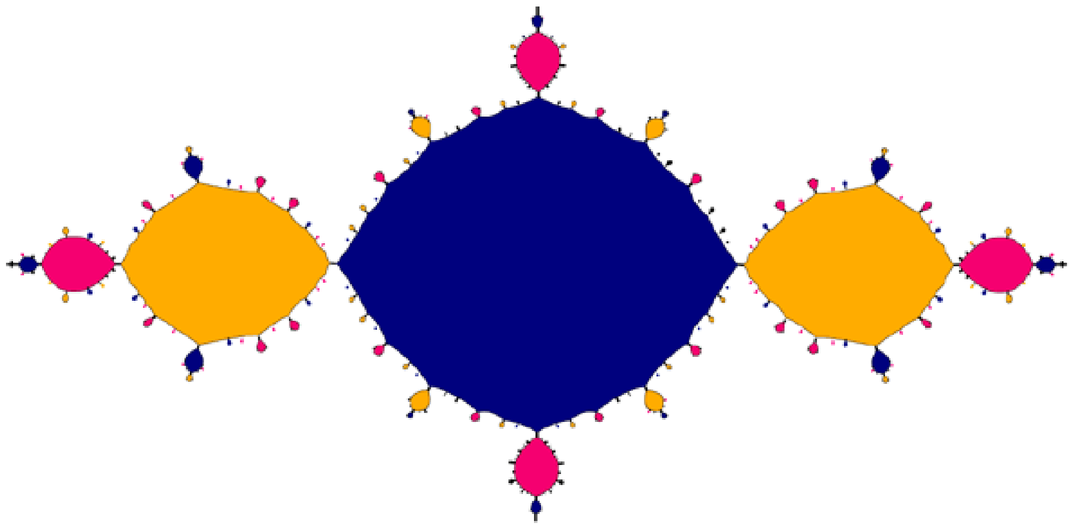
To determine if a complex number $z = a + bi$ belongs to the Julia set, repeatedly compute the values of

$$z^2 - 1,\ \left(z^2 - 1\right)^2 - 1,\ \left[\left(z^2 - 1\right)^2 - 1\right]^2 - 1,\ \ldots$$

DECIDING WHETHER A COMPLEX NUMBER IS IN THE JULIA SET (cont.)



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If the absolute values of any of the resulting complex numbers exceed 2, then the complex number z is not in the Julia set. Otherwise z is part of this set and the point (a, b) should be shaded in the graph.

Example 5 DECIDING WHETHER A COMPLEX NUMBER IS IN THE JULIA SET (cont.)

Determine whether each number belongs to the Julia set.

(a) $z = 0 + 0i$

$$z = 0 + 0i = 0$$

$$z^2 - 1 = 0^2 - 1 = -1$$

$$\left(z^2 - 1\right)^2 - 1 = (-1)^2 - 1 = 0$$

$$\left[\left(z^2 - 1\right)^2 - 1\right]^2 - 1 = 0^2 - 1 = -1$$

The calculations repeat as 0, –1, 0, –1, and so on.

The absolute values are either 0 or 1, which do not exceed 2, so 0 + 0*i* is in the Julia set, and the point (0, 0) is part of the graph.

▶ Example 5

DECIDING WHETHER A COMPLEX NUMBER IS IN THE JULIA SET (cont.)

(b) $z = 1 + 1i$

$$z^2 - 1 = (1 + 1i)^2 - 1 = \left(1 + 2i + i^2\right) - 1 = -1 + 2i$$

The absolute value is $\sqrt{(-1)^2 + 2^2} = \sqrt{5}.$

$\sqrt{5} > 2,$ so 1 + 1*i* is not in the Julia set and (1, 1) is not part of the graph.

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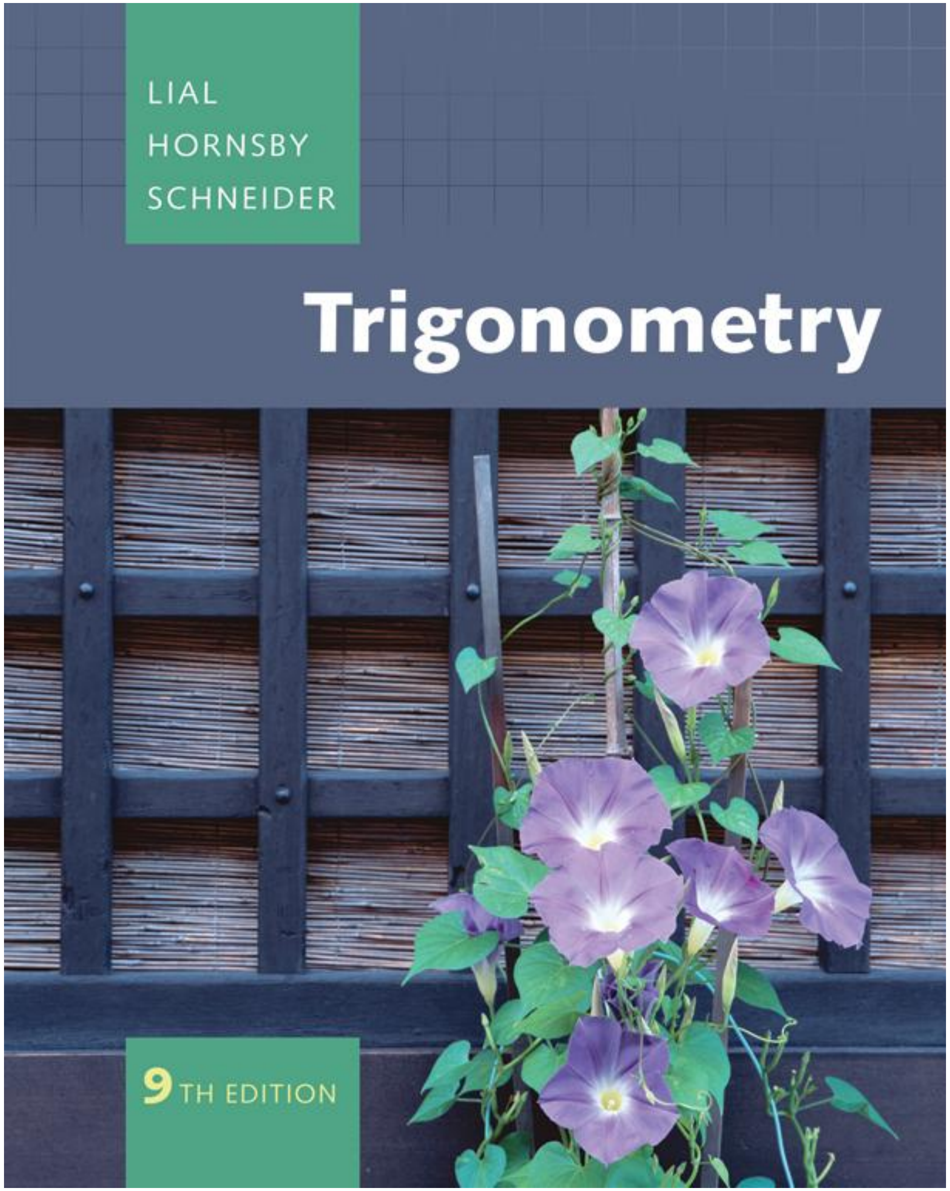
Complex Numbers, Polar Equations, and Parametric Equations



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8 Complex Numbers, Polar Equations, and Parametric Equations

8.3 The Product and Quotient Theorems

Products of Complex Numbers in Trigonometric Form ▪ Quotients of Complex Numbers in Trigonometric Form

Product Theorem

If $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ are any two complex numbers, then

$$\left[r_1(\cos\theta_1 + i\sin\theta_1)\right]\cdot\left[r_2(\cos\theta_2 + i\sin\theta_2)\right]$$
$$= r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right].$$

In compact form, this is written

$$(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}\theta_2) = r_1r_2 \operatorname{cis}(\theta_1+\theta_2).$$

Example 1 USING THE PRODUCT THEOREM

Find the product of 3(cos 45° + *i* sin 45°) and 2(cos 135° + *i* sin 135°). Write the result in rectangular form.

$$\left[3\left(\cos 45^\circ + i\sin 45^\circ\right)\right]\left[2\left(\cos 135^\circ + i\sin 135^\circ\right)\right]$$
$$= 3\cdot 2\left[\cos\left(45^\circ + 135^\circ\right) + i\sin\left(45^\circ + 135^\circ\right)\right]$$
$$= 6\left(\cos 180^\circ + i\sin 180^\circ\right)$$
$$= 6\left(-1 + i\cdot 0\right)$$
$$= -6$$

Quotient Theorem

If $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ are any two complex numbers, where $r_2(\cos\theta_2 + i\sin\theta_2) \neq 0,$ then

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

In compact form, this is written

$$\frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2).$$

Example 2 USING THE QUOTIENT THEOREM

Find the quotient $\dfrac{10\,\text{cis}(-60°)}{5\,\text{cis}\,150°}$. Write the result in rectangular form.

$$\frac{10\,\text{cis}(-60°)}{5\,\text{cis}\,150°} = \frac{10}{5}\text{cis}(-60° - 150°)$$

$$= 2\,\text{cis}(-210°)$$

$$= 2\left[\cos(-210°) + i\sin(-210°)\right]$$

$$= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$$

$$= -\sqrt{3} + i$$

▶ Example 3 USING THE PRODUCT AND QUOTIENT THEOREMS WITH A CALCULATOR

Use a calculator to find the following. Write the results in rectangular form.

(a) $(9.3\text{cis}125.2°)(2.7\text{cis}49.8°)$

$$= 9.3(2.7)\text{cis}(125.2° + 49.8°)$$

$$= 25.11\text{cis}175°$$

$$= 25.11(\cos 175° + i\sin 175°)$$

$$\approx 25.11[-.99619470 + i(.08715574)]$$

$$\approx -25.0144 + 2.1885i$$

Example 3

USING THE PRODUCT AND QUOTIENT THEOREMS WITH A CALCULATOR (continued)

(b) $$\frac{10.42\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)}{5.21\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$$

$$=\frac{10.42}{5.21}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{5}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{5}\right)\right]$$

$$=2\left(\cos\frac{11\pi}{20}+i\sin\frac{11\pi}{20}\right)$$

$$\approx -.3129+1.9754i$$

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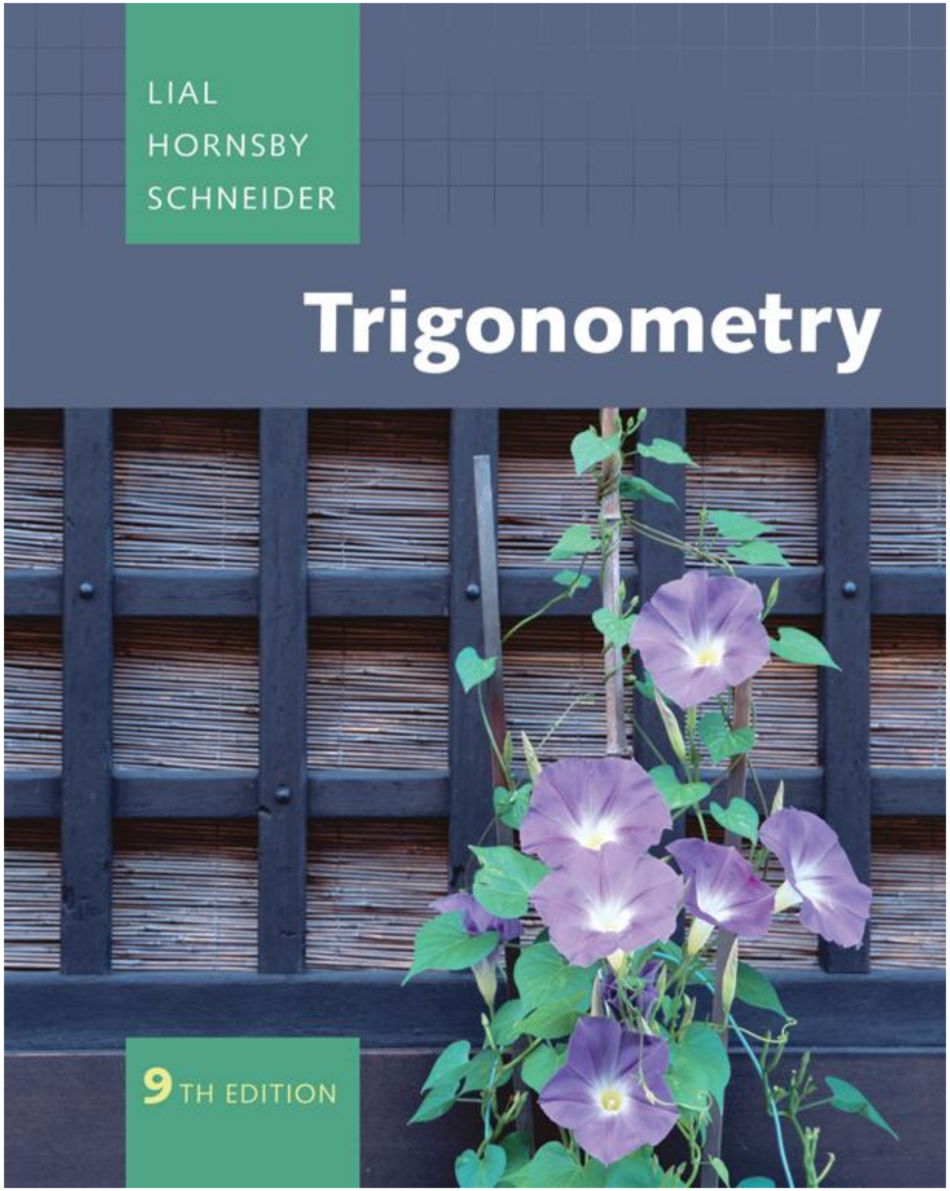
Complex Numbers, Polar Equations, and Parametric Equations



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8 Complex Numbers, Polar Equations, and Parametric Equations

8.4 De Moivre's Theorem; Powers and Roots of Complex Numbers

Powers of Complex Numbers (De Moivre's Theorem) ▪ Roots of Complex Numbers

De Moivre's Theorem

If $r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta).$$

In compact form, this is written

$$\left[r\,\text{cis}\,\theta\right]^n = r^n(\text{cis}\,n\theta).$$

Example 1

FINDING A POWER OF A COMPLEX NUMBER

Find $\left(1+i\sqrt{3}\right)^8$ and express the result in rectangular form.

First write $1+i\sqrt{3}$ in trigonometric form.

$$r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2 \text{ and } \tan\theta=\sqrt{3}.$$

Because *x* and *y* are both positive, θ is in quadrant I, so $\theta = 60$.

$$1+i\sqrt{3}=2\operatorname{cis}60°$$

Example 1 FINDING A POWER OF A COMPLEX NUMBER (continued)

Now apply De Moivre's theorem.

$$\left(1+i\sqrt{3}\right)^8 = \left[2\left(\cos 60^\circ + i\sin 60^\circ\right)\right]^8$$

$$= 2^8\left[\cos\left(8\cdot 60^\circ\right) + i\sin\left(8\cdot 60^\circ\right)\right]$$

$$= 256\left(\cos 480^\circ + i\sin 480^\circ\right)$$

$$= 256\left(\cos 120^\circ + i\sin 120^\circ\right)$$

480 and 120 are coterminal.

$$= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -128 + 128i\sqrt{3}$$ Rectangular form

*n*th Root

For a positive integer n, the complex number $a + bi$ is an ***n*th root** of the complex number $x + yi$ if

$$(a + bi)^n = x + yi.$$

*n*th Root Theorem

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\left(\cos\alpha + i\sin\alpha\right) \text{ or } \sqrt[n]{r}\,\text{cis}\,\alpha,$$

where

$$\alpha = \frac{\theta + 360^\circ \cdot k}{n}, \text{ or } \alpha = \frac{\theta}{n} + \frac{360^\circ \cdot k}{n},$$

$$k = 0, 1, 2, \ldots, n-1$$

Note

In the statement of the *n*th root theorem, if θ is in radians, then

$$\alpha = \frac{\theta + 2\pi k}{n} \quad \text{or} \quad \alpha = \frac{\theta}{n} + \frac{2\pi k}{n}.$$

Example 2 FINDING COMPLEX ROOTS

Find the two square roots of 4*i*. Write the roots in rectangular form.

Write 4*i* in trigonometric form: $4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$r = 4,\ \theta = \frac{\pi}{2}$$

The square roots have absolute value $\sqrt{4} = 2$ and argument

$$\alpha = \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2} = \frac{\pi}{4} + \pi k.$$

Example 2 FINDING COMPLEX ROOTS (continued)

Since there are two square roots, let *k* = 0 and 1.

$$\text{If } k = 0, \text{ then } \alpha = \frac{\pi}{4} + \pi \cdot 0 = \frac{\pi}{4}.$$

$$\text{If } k = 1, \text{ then } \alpha = \frac{\pi}{4} + \pi \cdot 1 = \frac{5\pi}{4}.$$

Using these values for α, the square roots are

$$2\,\text{cis}\,\frac{\pi}{4} \text{ and } 2\,\text{cis}\,\frac{5\pi}{4}.$$

Example 2 FINDING COMPLEX ROOTS (continued)

$$2\,\text{cis}\,\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$2\,\text{cis}\,\frac{\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} - i\sqrt{2}$$

Example 3 FINDING COMPLEX ROOTS

Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.

Write $-8+8i\sqrt{3}$ in trigonometric form:

$$r=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}=\sqrt{256}=16$$

$$\tan\theta=\frac{8\sqrt{3}}{-8}=-\sqrt{3}\Rightarrow\theta=120^\circ$$

$$-8+8i\sqrt{3}=16\operatorname{cis}120^\circ$$

The fourth roots have absolute value $\sqrt[4]{16}=2$ and argument

$$\alpha=\frac{120^\circ}{4}+\frac{360^\circ\cdot k}{4}=30^\circ+90^\circ\cdot k.$$

Example 3 FINDING COMPLEX ROOTS (continued)

Since there are four roots, let *k* = 0, 1, 2, and 3.

$$\text{If } k = 0, \text{ then } \alpha = 30° + 90° \cdot 0 = 30°.$$

$$\text{If } k = 1, \text{ then } \alpha = 30° + 90° \cdot 1 = 120°.$$

$$\text{If } k = 2, \text{ then } \alpha = 30° + 90° \cdot 2 = 210°.$$

$$\text{If } k = 3, \text{ then } \alpha = 30° + 90° \cdot 3 = 300°.$$

Using these values for *α*, the fourth roots are
2 cis 30 , 2 cis 120 , 2 cis 210 , and 2 cis 300 .

Example 3 FINDING COMPLEX ROOTS (continued)

$$2\text{ cis } 30° = 2(\cos 30° + i \sin 30°) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$$

$$2\text{ cis } 120° = 2(\cos 120° + i \sin 120°) = 2\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= -1 + i\sqrt{3}$$

$$2\text{ cis } 210° = 2(\cos 210° + i \sin 210°) = 2\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$

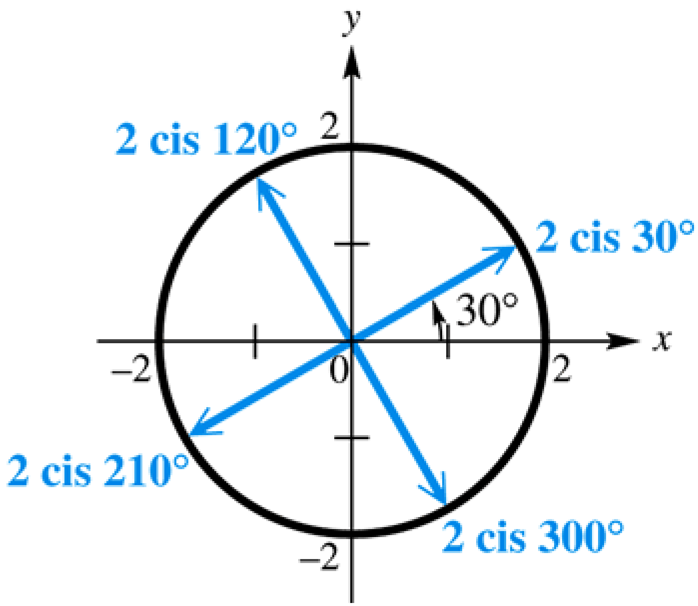
$$= -\sqrt{3} - i$$

$$2\text{ cis } 300° = 2(\cos 300° + i \sin 300°) = 2\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= 1 - i\sqrt{3}$$

Example 3 **FINDING COMPLEX ROOTS (continued)**

The graphs of the roots lie on a circle with center at the origin and radius 2. The roots are equally spaced about the circle, 90 apart.



Example 4 SOLVING AN EQUATION BY FINDING COMPLEX ROOTS

Find all complex number solutions of $x^5 - i = 0$. Graph them as vectors in the complex plane.

$$x^5 - 1 = 0 \Rightarrow x^5 = 1$$

There is one real solution, 1, while there are five complex solutions.

Write 1 in trigonometric form:

$$1 = 1 + 0i \Rightarrow r = 1 \text{ and } \tan\theta = 0 \Rightarrow \theta = 0^\circ$$

$$1 = 1(\cos 0^\circ + i \sin 0^\circ)$$

Example 4 **SOLVING AN EQUATION BY FINDING COMPLEX ROOTS (continued)**

The fifth roots have absolute value $\sqrt[5]{1} = 1$ and argument

$$\alpha = \frac{0°}{5} + \frac{360° \cdot k}{5} = 0° + 72° \cdot k = 72° \cdot k.$$

Since there are five roots, let *k* = 0, 1, 2, 3, and 4.

$$\text{If } k = 0, \text{ then } \alpha = 72° \cdot 0 = 0°.$$

$$\text{If } k = 1, \text{ then } \alpha = 72° \cdot 1 = 72°.$$

$$\text{If } k = 2, \text{ then } \alpha = 72° \cdot 2 = 144°.$$

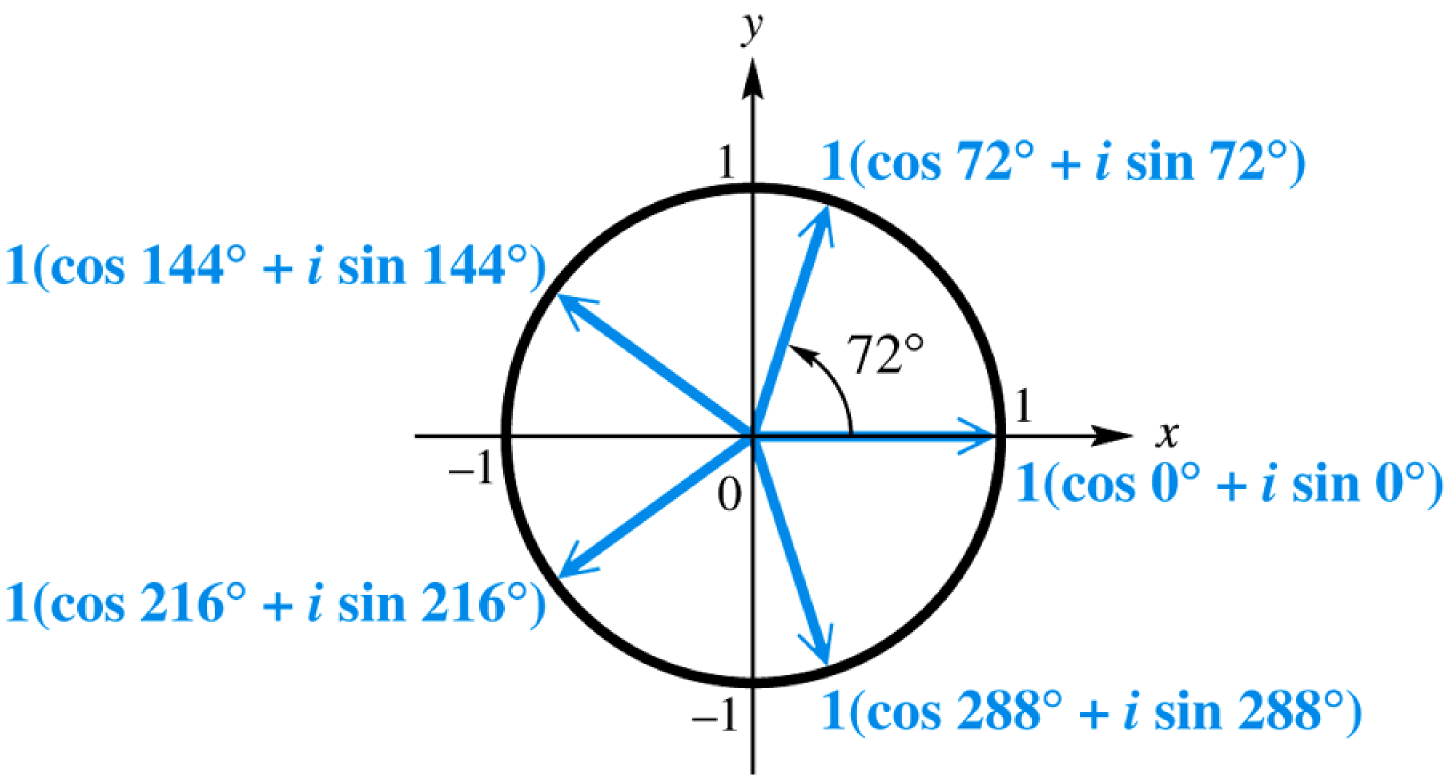
$$\text{If } k = 3, \text{ then } \alpha = 72° \cdot 3 = 216°.$$

$$\text{If } k = 4, \text{ then } \alpha = 72° \cdot 4 = 288°.$$

Solution set: {cis 0 , cis 72 , cis 144 , cis 216 , cis 288 }

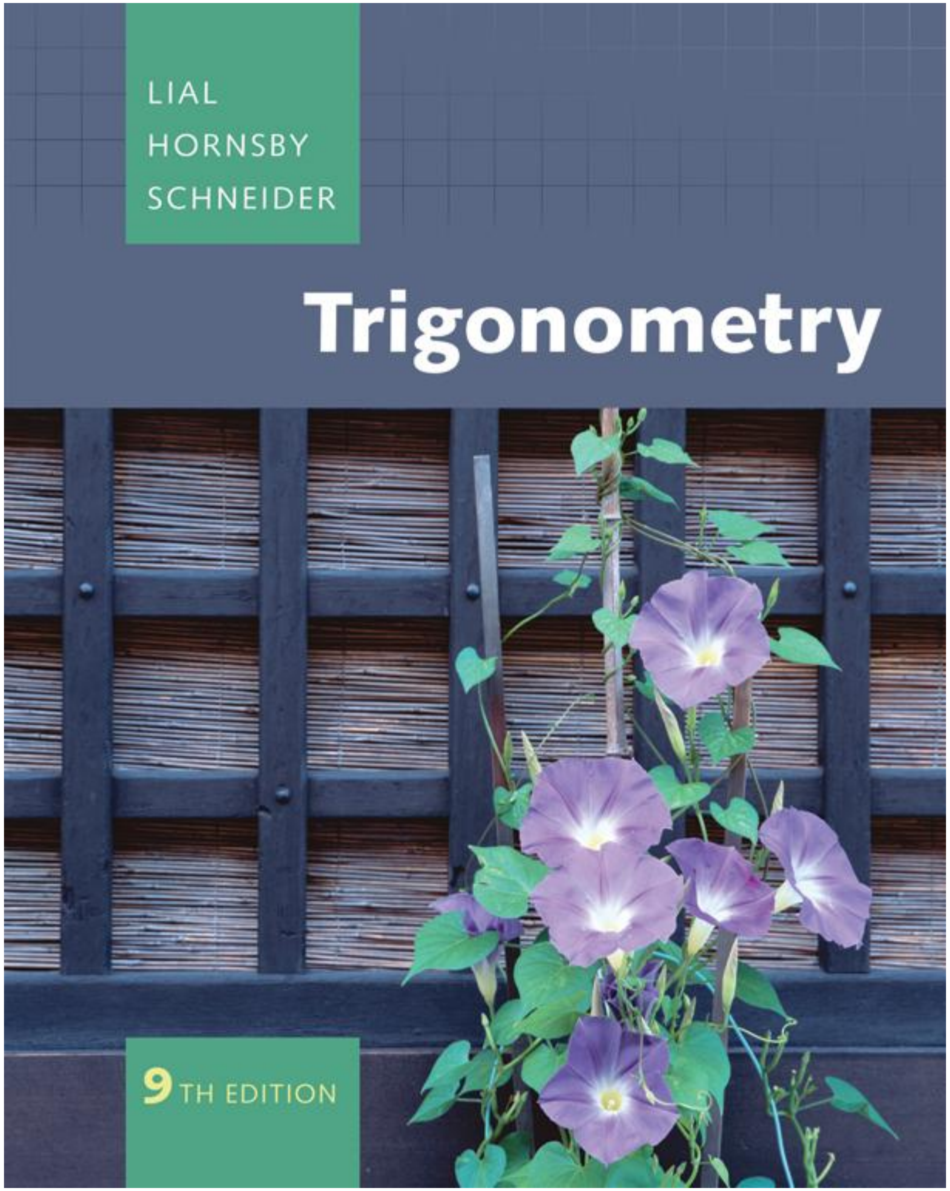
▶ Example 4 SOLVING AN EQUATION BY FINDING COMPLEX ROOTS (continued)

The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72 apart.



8

Complex Numbers, Polar Equations, and Parametric Equations



PEARSON Education

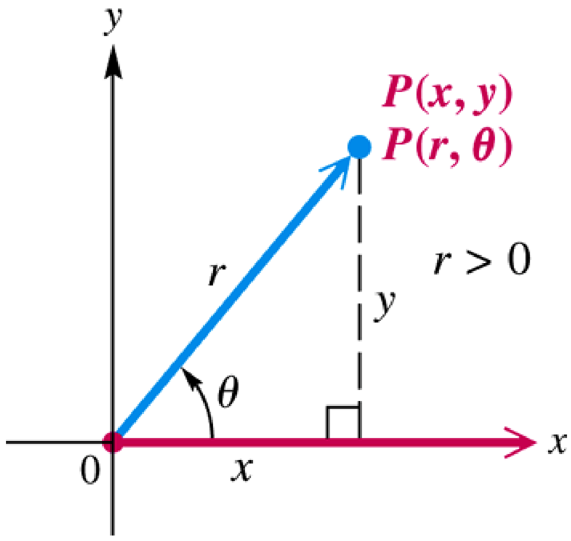
8 Complex Numbers, Polar Equations, and Parametric Equations

8.5 Polar Equations and Graphs

Polar Coordinate System ▪ Graphs of Polar Equations ▪ Converting from Polar to Rectangular Equations ▪ Classifying Polar Equations

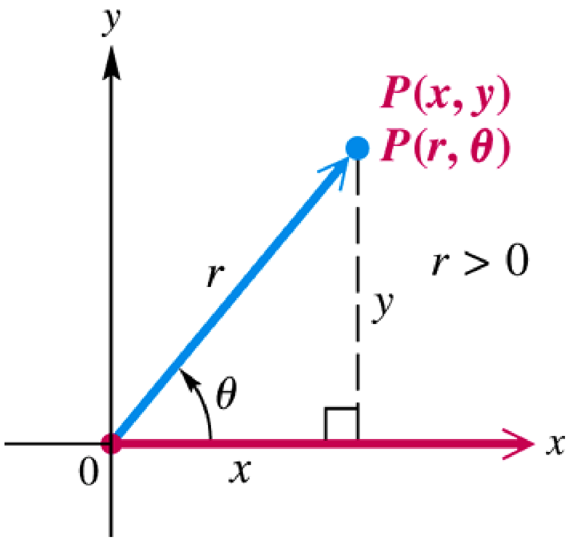
Polar Coordinate System

The **polar coordinate system** is based on a point, called the **pole**, and a ray, called the **polar axis**.



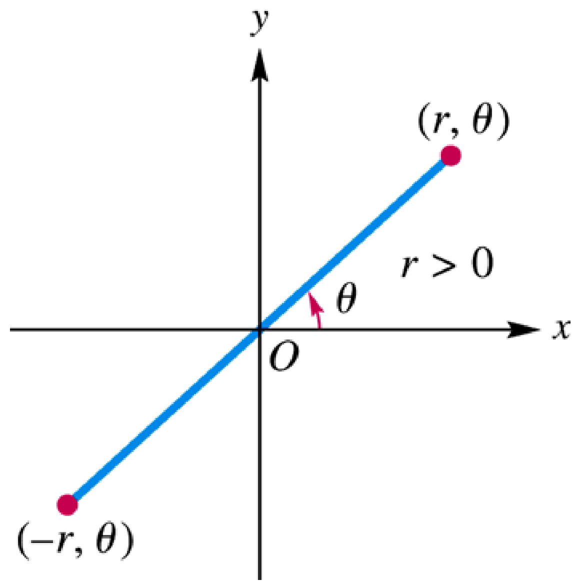
Polar Coordinate System

- Point *P* has rectangular coordinates (*x*, *y*).
- Point *P* can also be located by giving the directed angle *θ* from the positive *x*-axis to ray *OP* and the directed distance *r* from the pole to point *P.*
- The **polar coordinates** of point *P* are (*r*, *θ*).



Polar Coordinate System

- If $r > 0$, then point P lies on the terminal side of θ.
- If $r < 0$, then point P lies on the ray pointing in the opposite direction of the terminal side of θ, a distance $|r|$ from the pole.



Rectangular and Polar Coordinates

If a point has rectangular coordinates (x, y) and polar coordinates (r, θ), then these coordinates are related as follows.

$$x = r\cos\theta \qquad y = r\sin\theta$$

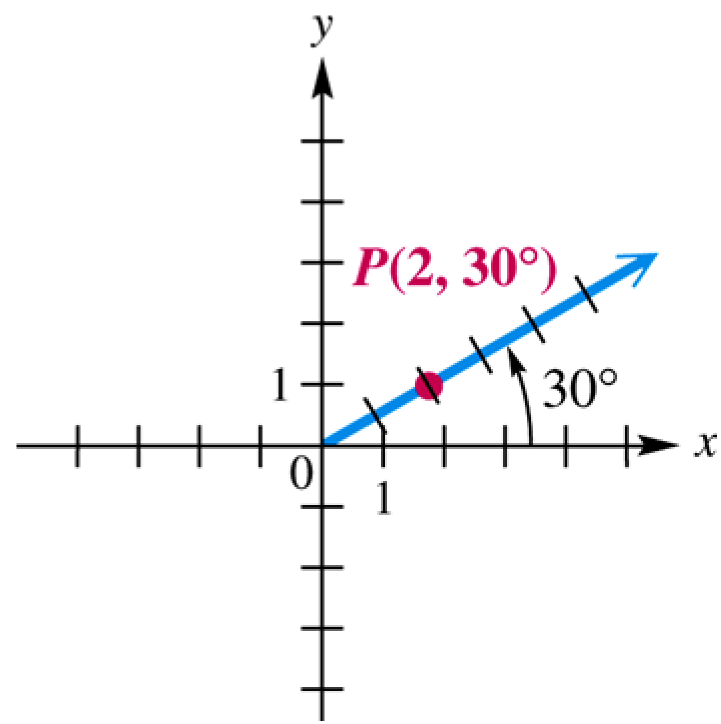
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}, \text{ if } x \neq 0$$

Example 1 PLOTTING POINTS WITH POLAR COORDINATES

Plot each point by hand in the polar coordinate system. Then, determine the rectangular coordinates of each point.

(a) $P(2, 30\ )$

$r = 2$ and $\theta = 30$, so point P is located 2 units from the origin in the positive direction making a 30 angle with the polar axis.



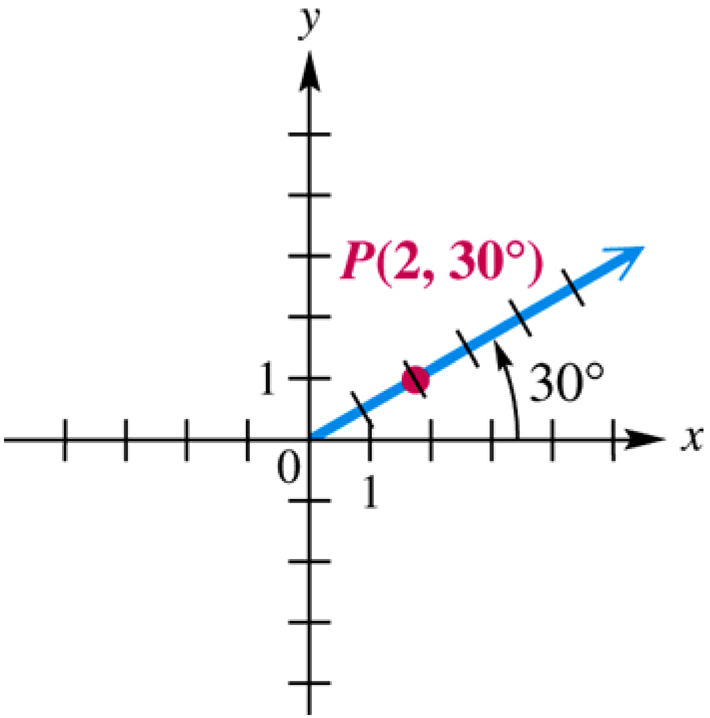
Example 1 PLOTTING POINTS WITH POLAR COORDINATES (continued)

Using the conversion formulas:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x = 2\cos 30^\circ \qquad y = 2\sin 30^\circ$$

$$x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \qquad y = 2\left(\frac{1}{2}\right) = 1$$

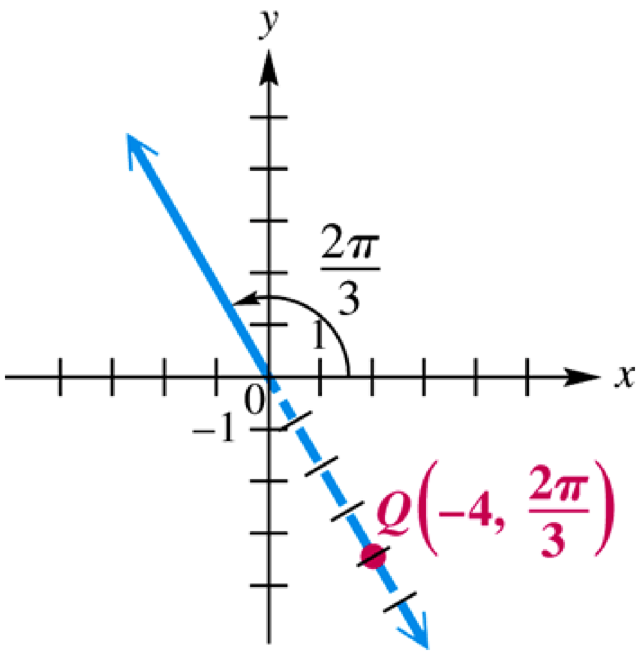


The rectangular coordinates are $\left(\sqrt{3}, 1\right)$.

Example 1 PLOTTING POINTS WITH POLAR COORDINATES (continued)

(b) $Q\left(-4, \frac{2\pi}{3}\right)$

Since r is negative, Q is 4 units in the opposite direction from the pole on an extension of the $\frac{2\pi}{3}$ ray.



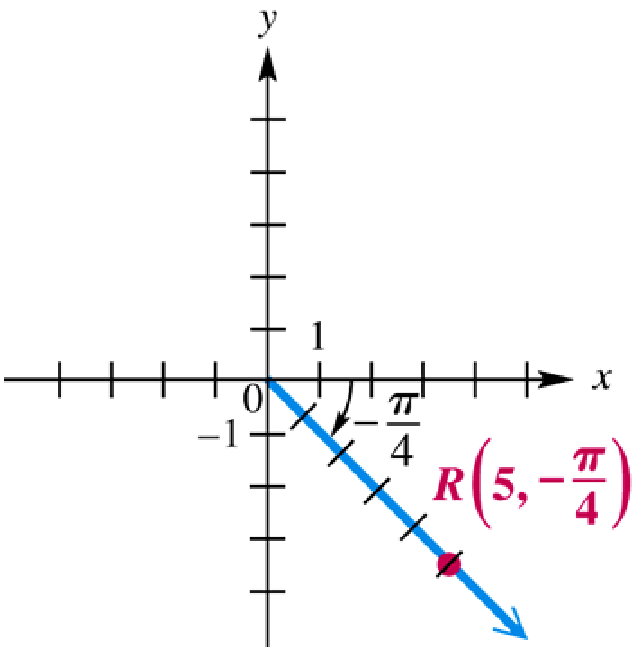
$$x = -4\cos\frac{2\pi}{3} = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\frac{2\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

The rectangular coordinates are $\left(2, -2\sqrt{3}\right)$.

Example 1 PLOTTING POINTS WITH POLAR COORDINATES (continued)

(c) $R\left(5, -\frac{\pi}{4}\right)$



Since θ is negative, the angle is measured in the clockwise direction.

$$x = 5\cos\left(-\frac{\pi}{4}\right) = 5\cos\left(\frac{7\pi}{4}\right) = \frac{5\sqrt{2}}{2}$$

$$x = 5\sin\left(-\frac{\pi}{4}\right) = 5\sin\left(\frac{7\pi}{4}\right) = -\frac{5\sqrt{2}}{2}$$

The rectangular coordinates are $\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$.

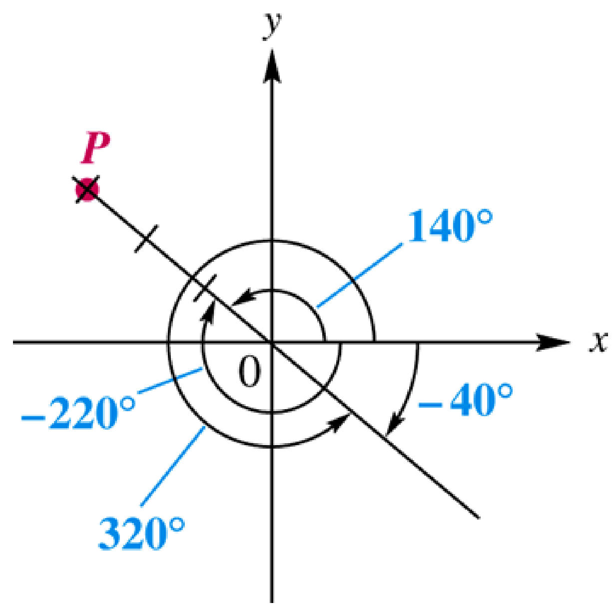
▶ Note

While a given point in the plane can have only one pair of rectangular coordinates, this same point can have an infinite number of pairs of polar coordinates.

Example 2 GIVING ALTERNATIVE FORMS FOR COORDINATES OF A POINT

(a) Give three other pairs of polar coordinates for the point *P*(3, 140).

Three pairs of polar coordinates for the point *P*(3, 140) are (3, –220), (−3, 320), and (−3, −40).



Example 2 — GIVING ALTERNATIVE FORMS FOR COORDINATES OF A POINT (continued)

(b) Determine two pairs of polar coordinates for the point with the rectangular coordinates (–1, 1).

The point (–1, 1) lies in quadrant II.

Since $\tan\theta = \dfrac{1}{-1} = -1$, one possible value for θ is 135 .

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Two pairs of polar coordinates are $\left(\sqrt{2}, 135°\right)$ and $\left(-\sqrt{2}, 315°\right)$.

Graphs of Polar Equations

An equation in which *r* and *θ* are the variables is a **polar equation**.

Derive the polar equation of the line $ax + by = c$ as follows:

$$ax + by = c$$

$$a(r\cos\theta) + b(r\sin\theta) = c$$

Convert from rectangular to polar coordinates.

$$r(a\cos\theta + b\sin\theta) = c$$

Factor out *r*.

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

General form for the polar equation of a line

Graphs of Polar Equations

Derive the polar equation of the circle $x^2 + y^2 = a^2$ as follows:

$$x^2 + y^2 = a^2$$

$$r^2 = a^2 \qquad r^2 = x^2 + y^2$$

$$r = \pm a$$

General form for the polar equation of a circle

Example 3

EXAMINING POLAR AND RECTANGULAR EQUATION OF LINES AND CIRCLES

For each rectangular equation, give the equivalent polar equation and sketch its graph.

(a) $y = x - 3$

In standard form, the equation is $x - y = 3$, so $a = 1$, $b = -1$, and $c = 3$.

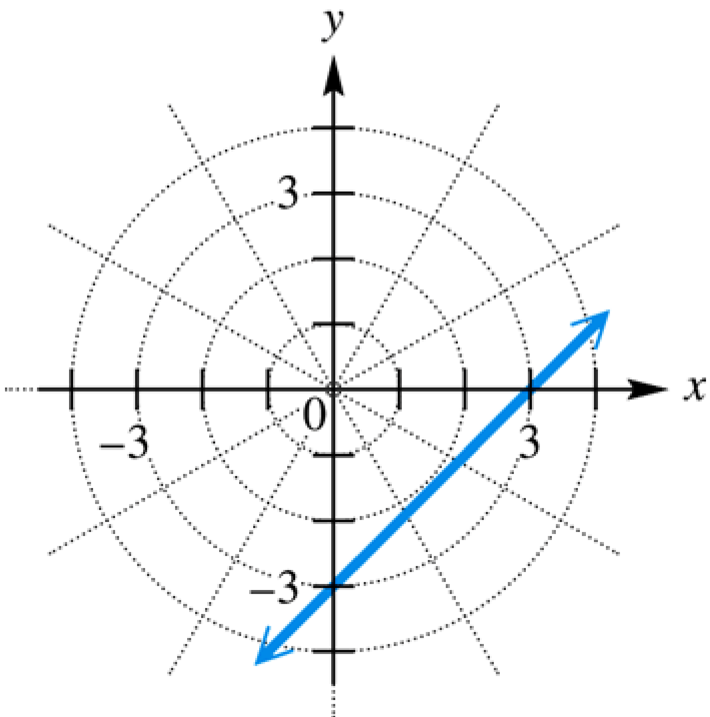
The general form for the polar equation of a line is

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

$y = x - 3$ is equivalent to $r = \dfrac{3}{\cos\theta - \sin\theta}$

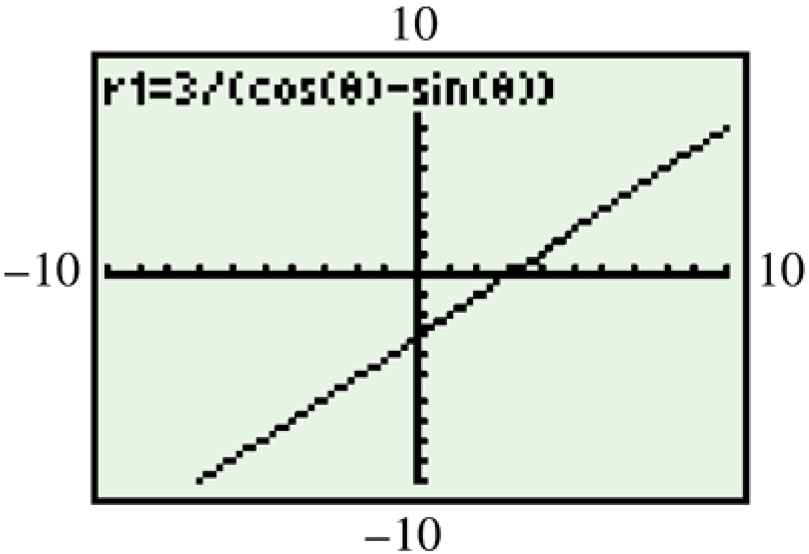
Example 3

EXAMINING POLAR AND RECTANGULAR EQUATION OF LINES AND CIRCLES (continued)



$y = x - 3$ (rectangular)

$$r = \frac{3}{\cos\theta - \sin\theta} \text{ (polar)}$$



Polar graphing mode

Example 3 EXAMINING POLAR AND RECTANGULAR EQUATION OF LINES AND CIRCLES (continued)

(b) $x^2 + y^2 = 4$

This is the graph of a circle with center at the origin and radius 2.

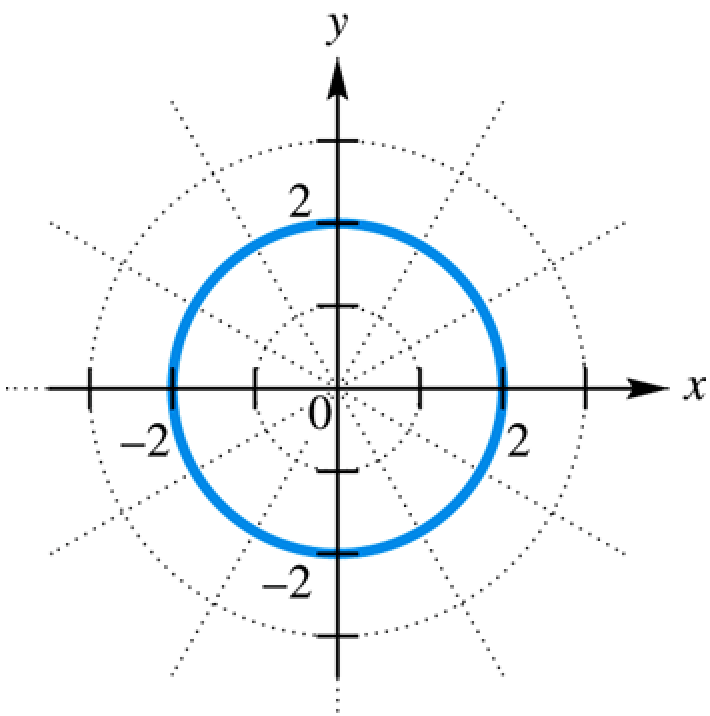
$$x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$$

Note that in polar coordinates it is possible for $r < 0$.

$x^2 + y^2 = 4$ is equivalent to $r = 2$ or $r = -2$.

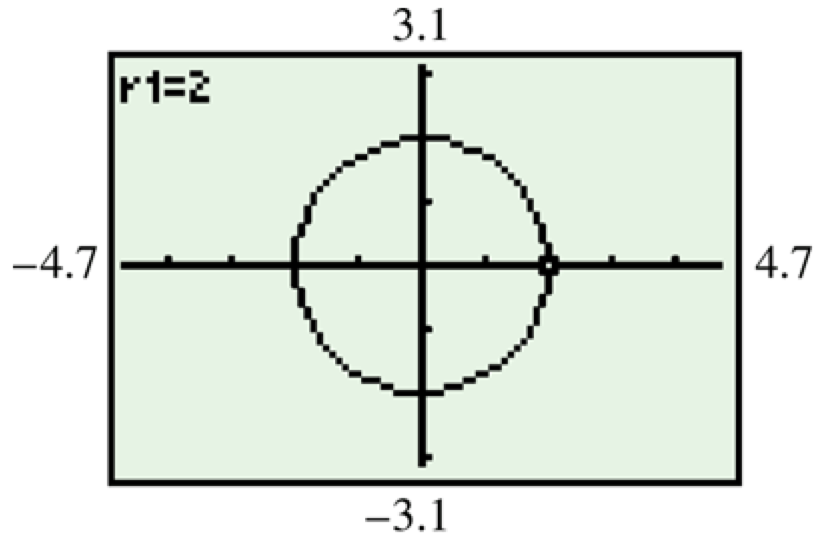
Example 3

EXAMINING POLAR AND RECTANGULAR EQUATION OF LINES AND CIRCLES (continued)



$x^2 + y^2 = 4$ (rectangular)

$r = 2$ (polar)



Polar graphing mode

Example 4 GRAPHING A POLAR EQUATION (CARDIOID)

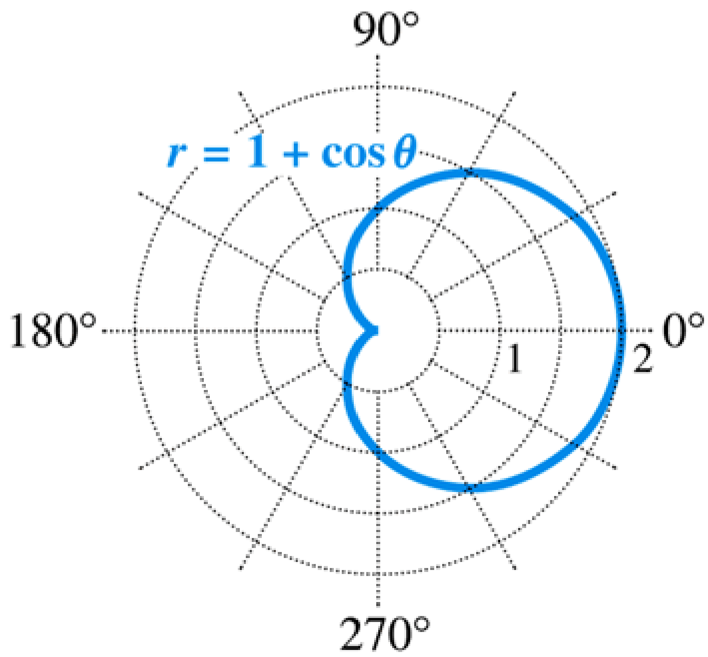
Graph $r = 1 + \cos\theta$.

Find some ordered pairs to determine a pattern of values of *r*.

θ	0°	30°	45°	60°	90°	120°
$\cos\theta$	1	.9	.7	.5	0	−.5
$r = 1 + \cos\theta$	2	1.9	1.7	1.5	1	.5
θ	135°	150°	180°	270°	315°	330°
$\cos\theta$	−.7	−.9	−1	0	.7	.9
$r = 1 + \cos\theta$	.3	.1	0	1	1.7	1.9

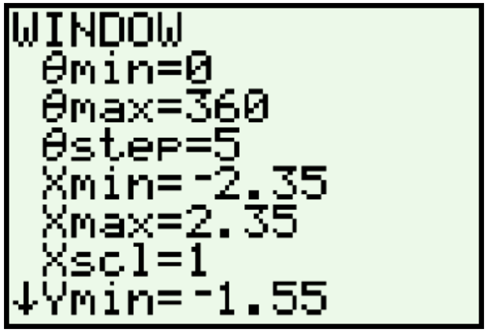
Example 4 GRAPHING A POLAR EQUATION (CARDIOID)

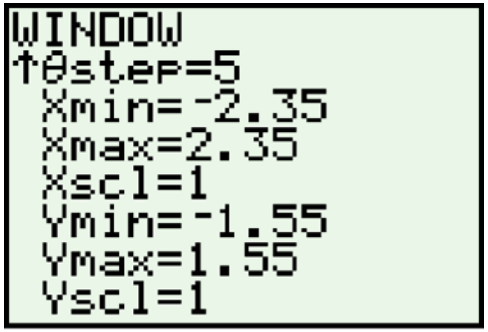
Connect the points in order from (2, 0) to (1.9, 30) to (1.7, 48) and so on.



▶ Example 4 GRAPHING A POLAR EQUATION (CARDIOD) (continued)

Choose degree mode and graph values of θ in the interval [0 , 360].





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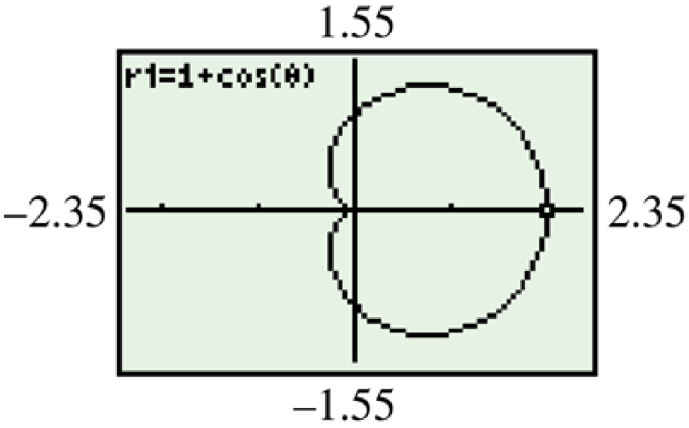


Example 5 GRAPHING A POLAR EQUATION (ROSE)

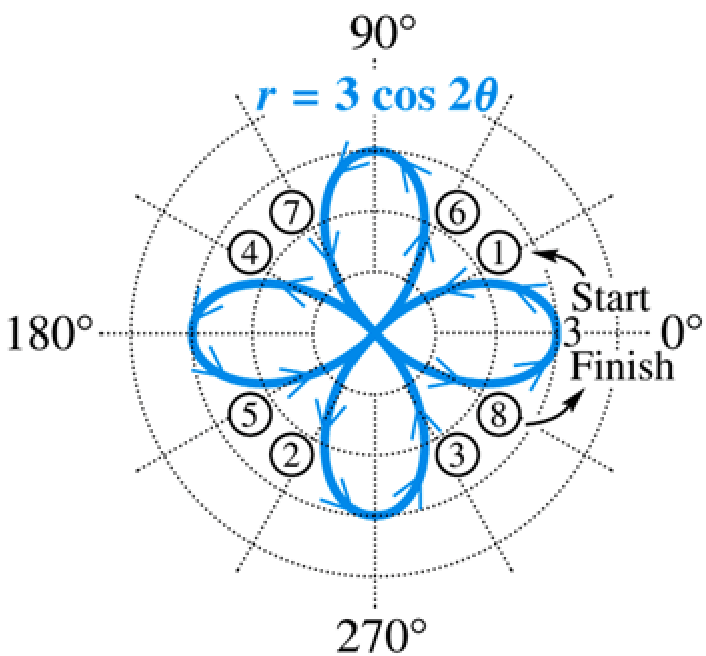
Graph $r = 3\cos 2\theta$.

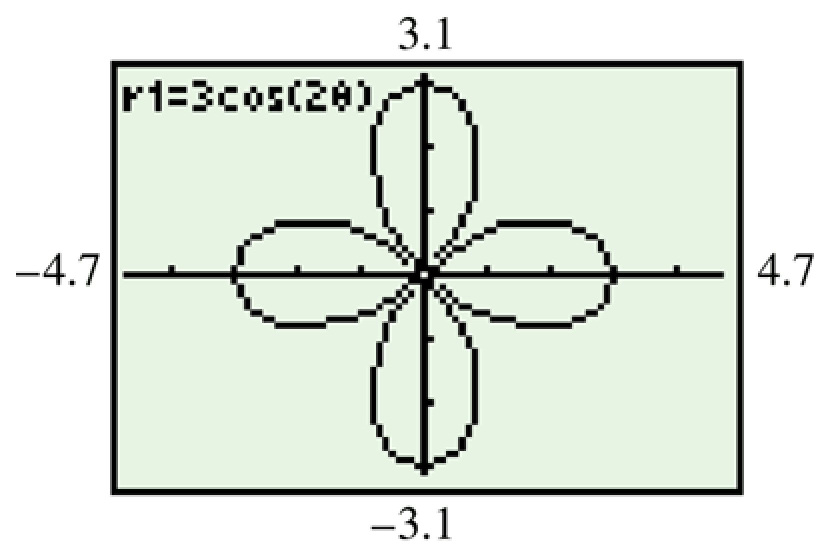
Find some ordered pairs to determine a pattern of values of *r*.

θ	0°	15°	30°	45°	60°	75°
2θ	0°	30°	60°	90°	120°	150°
$\cos 2\theta$	1	.9	.5	0	−.5	−.9
$r = 3\cos 2\theta$	3	2.6	1.5	0	−1.5	−2.6
θ	90°	105°	120°	135°	150°	165°
2θ	180°	210°	240°	270°	300°	330°
$\cos 2\theta$	−1	−.9	−.5	0	.5	.9
$r = 3\cos 2\theta$	−3	−2.6	−1.5	0	1.5	2.6

Example 5 GRAPHING A POLAR EQUATION (ROSE) (continued)

Connect the points in order from (3, 0) to (2.6, 15) to (1.5, 30) and so on. Notice how the graph is developed with a continuous curve, starting with the upper half of the right horizontal leaf and ending with the lower half of that leaf.





Polar graphing mode

Example 6 GRAPHING A POLAR EQUATION (LEMNISCATE)

Graph $r^2 = \cos 2\theta$.

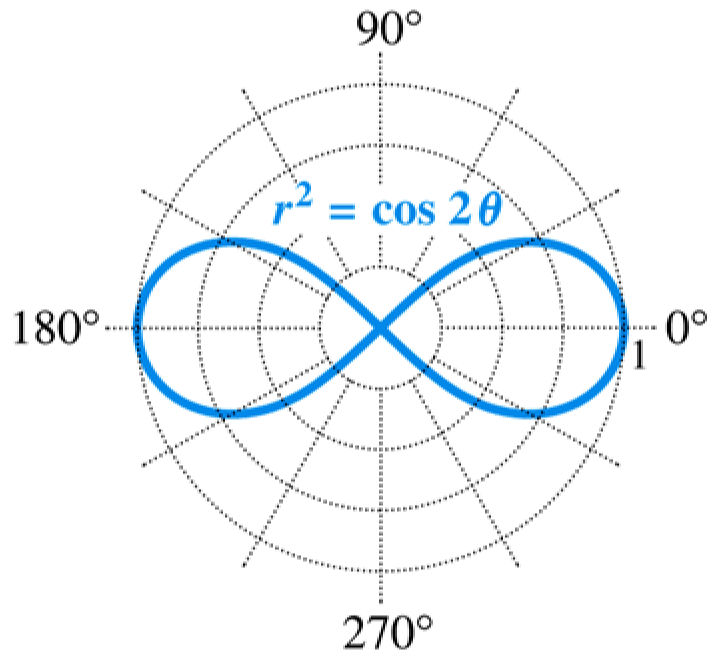
Find some ordered pairs to determine a pattern of values of *r*.

Values of θ for $45 \leq \theta \leq 135$ are not included in the table because the corresponding values of 2θ are negative. Values of θ larger than 180 give 2θ larger than 360 and would repeat the values already found.

θ	0°	30°	45°	135°	150°	180°
2θ	0°	60°	90°	270°	300°	360°
$\cos 2\theta$	1	.5	0	0	.5	1
$r = \pm\sqrt{\cos 2\theta}$	±1	±.7	0	0	±.7	±1

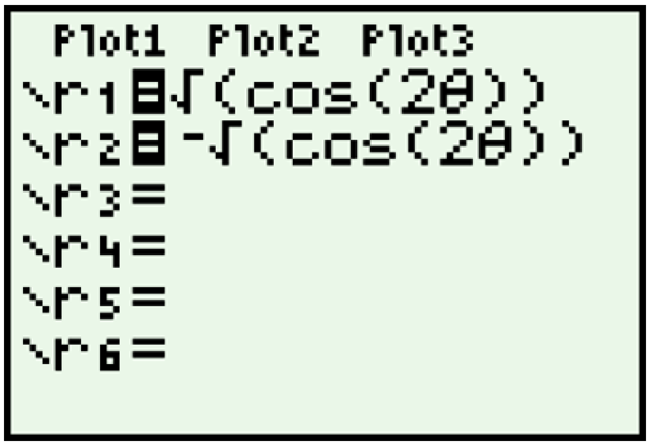
Example 6

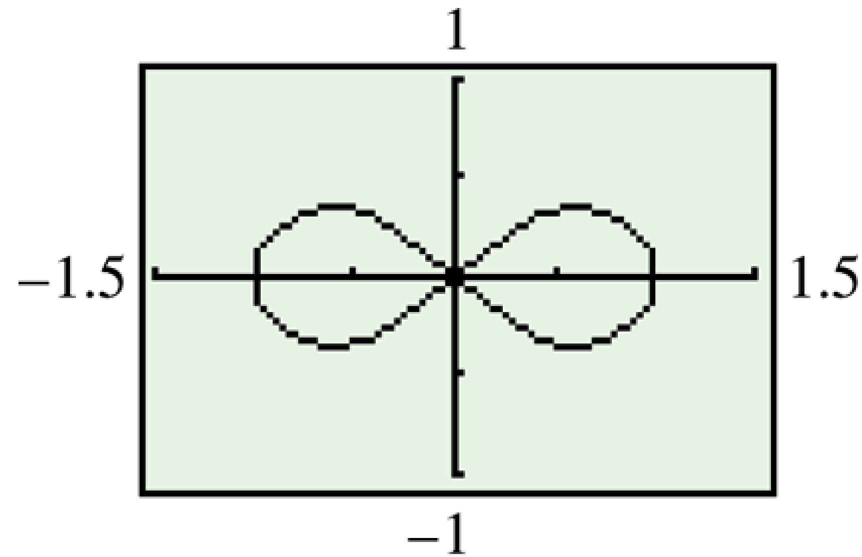
GRAPHING A POLAR EQUATION (LEMNISCATE) (continued)



▶ Example 6 GRAPHING A POLAR EQUATION (LEMNISCATE) (continued)

To graph $r^2 = \cos 2\theta$ with a graphing calculator, let $r_1 = \sqrt{\cos 2\theta}$ and $r_2 = -\sqrt{\cos 2\theta}$.





▶ Example 7

GRAPHING A POLAR EQUATION (SPIRAL OF ARCHIMEDES)

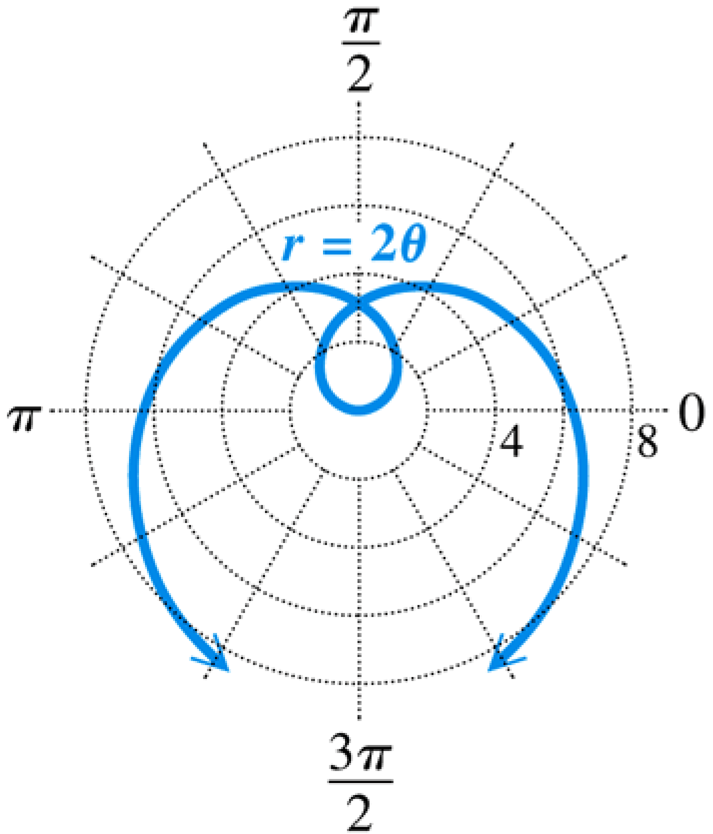
Graph $r = 2\theta$, (θ measured in radians).

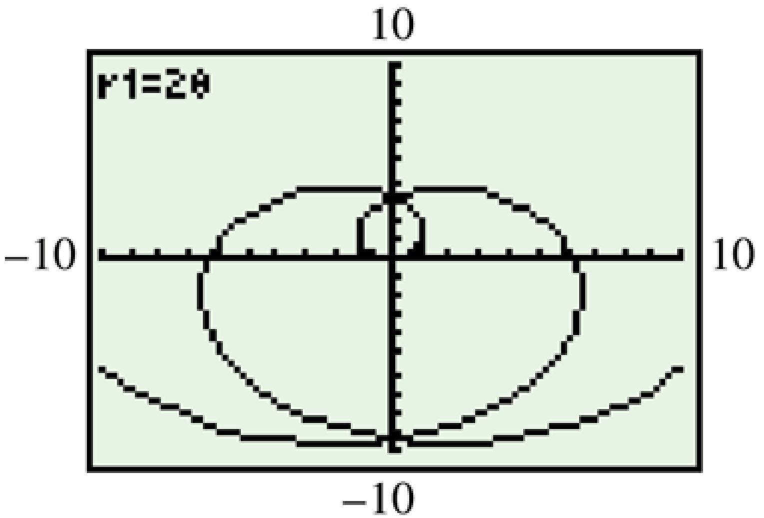
Find some ordered pairs to determine a pattern of values of *r*.

Since $r = 2\theta$, also consider negative values of θ.

θ	$-\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$r = 2\theta$	−6.3	−3.1	−1.6	0	1	2.1	3.1	6.3	9.4	12.6

Example 7 GRAPHING A POLAR EQUATION (SPIRAL OF ARCHIMEDES) (continued)





$-2\pi \leq \theta \leq 2\pi$

More of the spiral can be seen in this calculator graph.

Example 8 CONVERTING A POLAR EQUATION TO A RECTANGULAR EQUATION

Convert the equation $r = \dfrac{4}{1+\sin\theta}$ to rectangular coordinates and graph.

$$r = \frac{4}{1+\sin\theta}$$

$$r + r\sin\theta = 4$$

Multiply both sides by 1 + sin θ.

$$\sqrt{x^2+y^2} + y = 4$$

$r = \sqrt{x^2+y^2}$, $y = r\sin\theta$

$$\sqrt{x^2+y^2} = 4 - y$$

$$x^2 + y^2 = (4-y)^2$$

Square both sides.

$$x^2 + y^2 = 16 - 8y + y^2$$

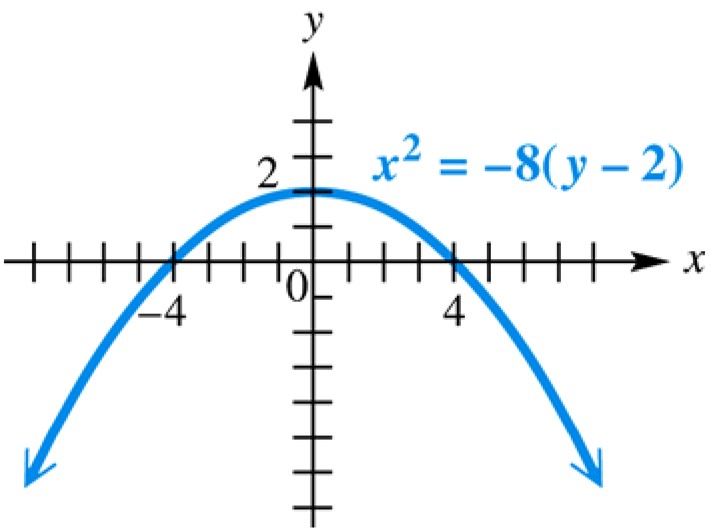
Expand.

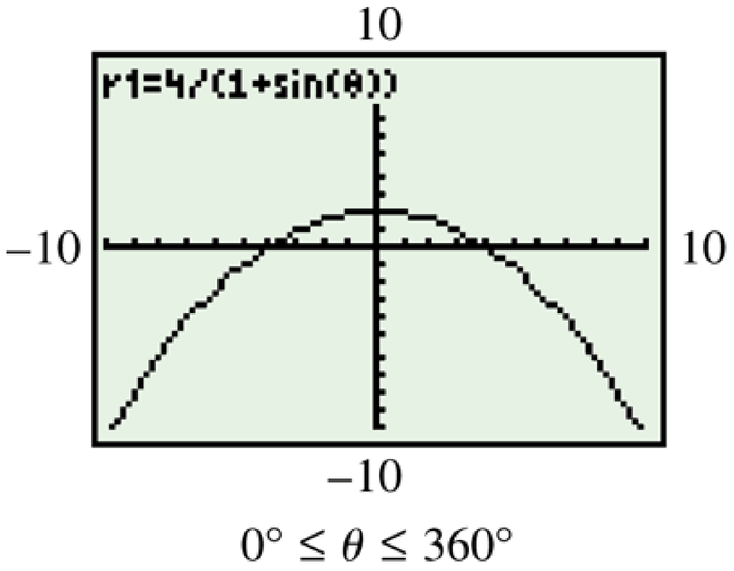
$$x^2 = -8y + 16$$

$$x^2 = -8(y-2)$$

Rectangular form

▶ Example 8 CONVERTING A POLAR EQUATION TO A RECTANGULAR EQUATION (cont.)

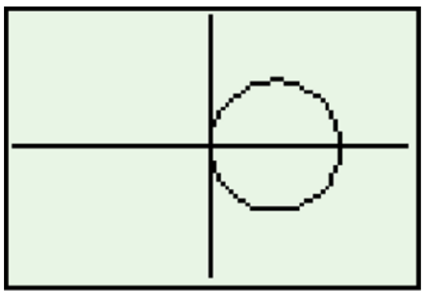




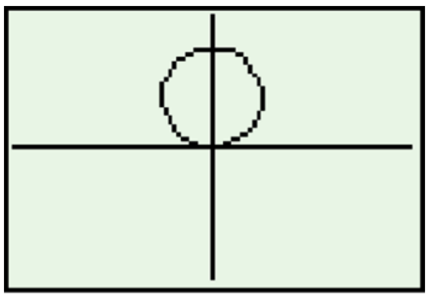
The graph is plotted with the calculator in polar mode.

Classifying Polar Equations

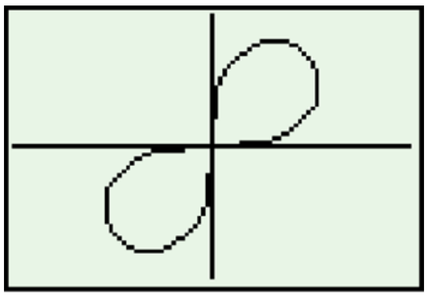
Circles and Lemniscates



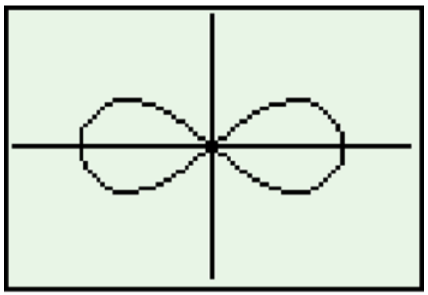
$r = a \cos \theta$



$r = a \sin \theta$



$r^2 = a^2 \sin 2\theta$

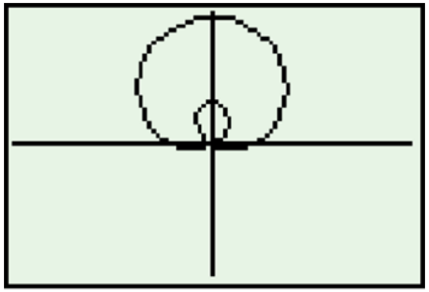


$r^2 = a^2 \cos 2\theta$

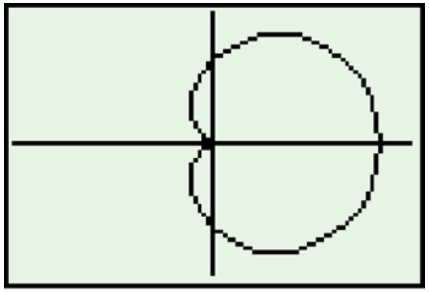
Classifying Polar Equations

Limaçons

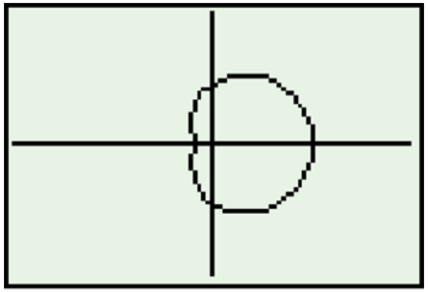
$$r = a \pm b\sin\theta \text{ or } r = a \pm b\cos\theta$$



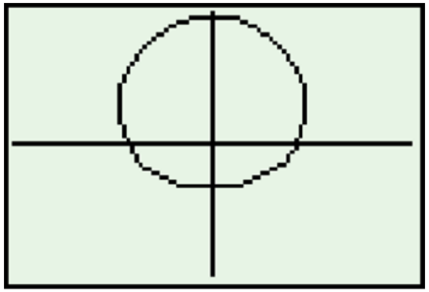
$\frac{a}{b} < 1$



$\frac{a}{b} = 1$



$1 < \frac{a}{b} < 2$



$\frac{a}{b} \geq 2$

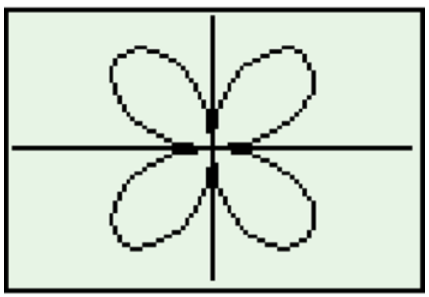
Classifying Polar Equations

Rose Curves

$2n$ leaves if n is even, $n \geq 2$

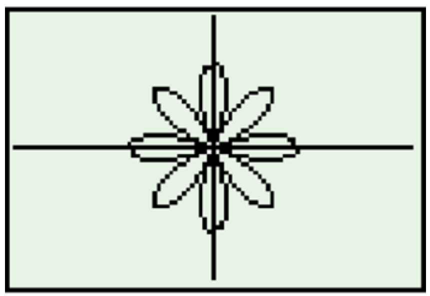
n leaves if n is odd

$n = 2$



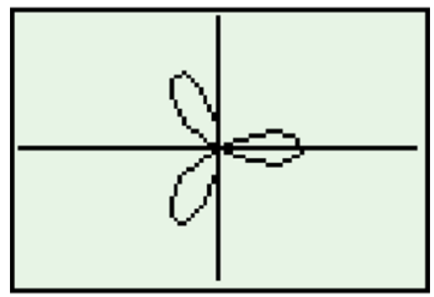
$r = a \sin n\theta$

$n = 4$



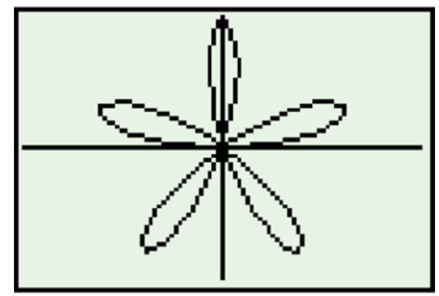
$r = a \cos n\theta$

$n = 3$



$r = a \cos n\theta$

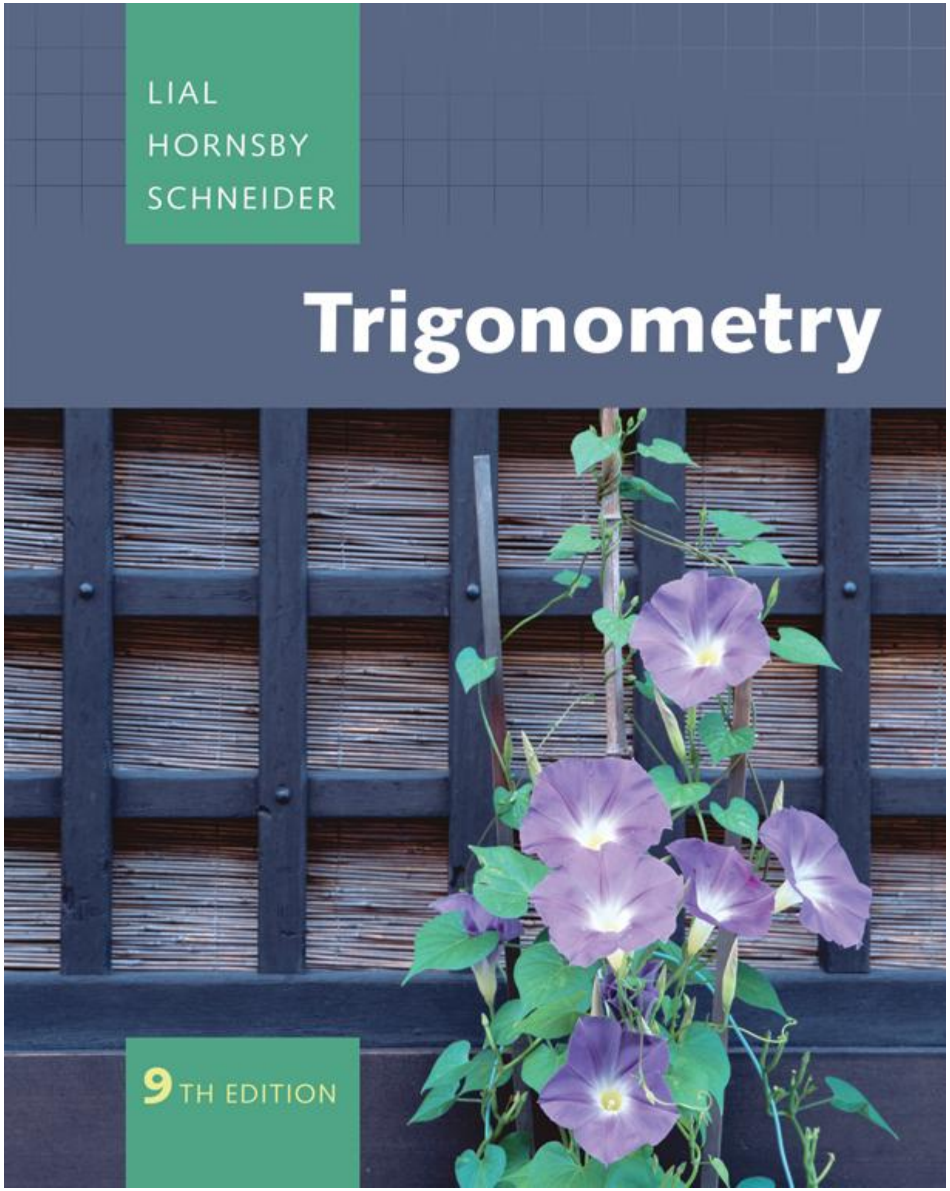
$n = 5$



$r = a \sin n\theta$

8

Complex Numbers, Polar Equations, and Parametric Equations



PEARSON Education

8 Complex Numbers, Polar Equations, and Parametric Equations

8.6 Parametric Equations, Graphs, and Applications

Basic Concepts ▪ Parametric Graphs and Their Rectangular Equivalents ▪ The Cycloid ▪ Applications of Parametric Equations

Parametric Equations of a Plane Curve

A **plane curve** is a set of points (x, y) such that $x = f(t)$, $y = g(t)$, and f and g are both defined on an interval I.

The equations $x = f(t)$ and $y = g(t)$ are **parametric equations** with **parameter *t***.

▶ Example 1

GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY

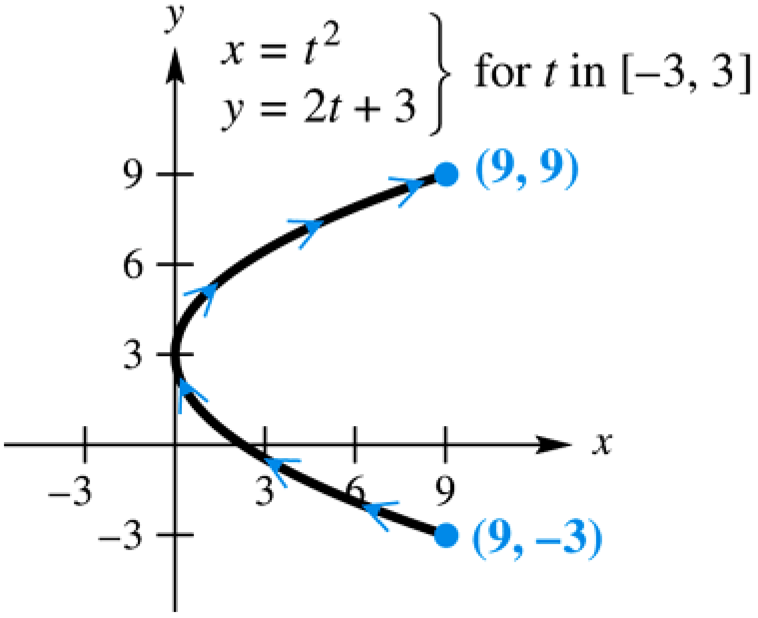
Let $x = t^2$ and $y = 2t + 3$ for t in [–3, 3]. Graph the set of ordered pairs (x, y).

Make a table of corresponding values of t, x, and y over the domain of t. Then plot the points.

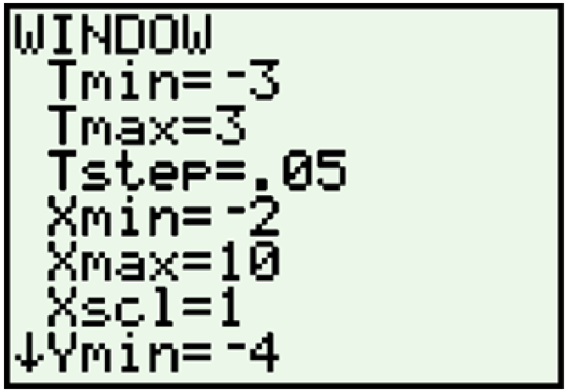
t	$x = t^2$	$y = 2t + 3$
−3	9	−3
−2	4	−1
−1	1	1
0	0	3
1	1	5
2	2	7
3	9	9

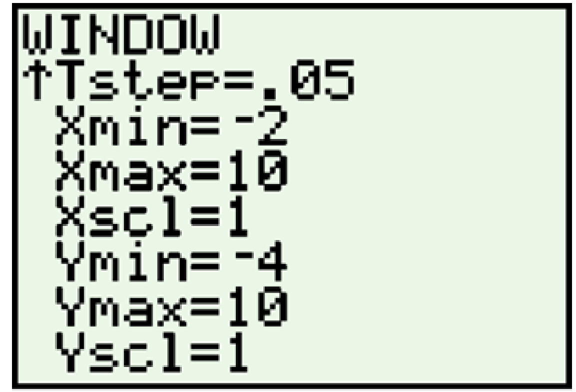
▶ Example 1 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY (continued)

The arrowheads indicate the direction the curve traces as *t* increases.

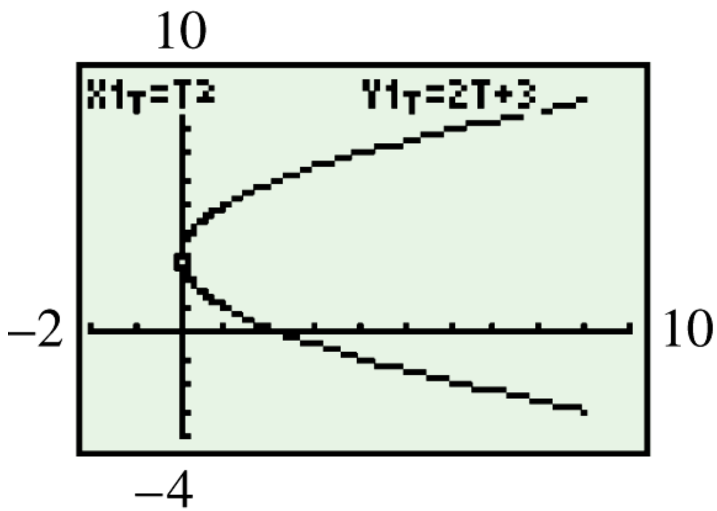


Example 1 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY (continued)





This is a continuation of the previous screen.



▶ Example 2 FINDING AN EQUIVALENT RECTANGULAR EQUATION

Find a rectangular equation for the plane curve defined as $x = t^2$ and $y = 2t + 3$ for t in [–3, 3]. (Example 1)

To eliminate the parameter *t*, solve either equation for *t.*

Since $y = 2t + 3$ leads to a unique solution for *t*, choose that equation.

$$y = 2t + 3 \Rightarrow t = \frac{y-3}{2}$$

Then $x = t^2 \Rightarrow x = \left(\frac{y-3}{2}\right)^2 = \frac{(y-3)^2}{4}$ or $4x = (y-3)^2$.

The rectangular equation is $4x = (y-3)^2$, for x in [0, 9].

Example 3 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY

Graph the plane curve defined by $x = 2 \sin t$, $y = 3 \cos t$, for t in $[0, 2\pi]$.

$$x = 2\sin t \Rightarrow \sin t = \frac{x}{2} \text{ and } y = 3\cos t \Rightarrow \cos t = \frac{y}{3}.$$

$$\sin^2 t + \cos^2 t = 1 \quad \text{Identity}$$

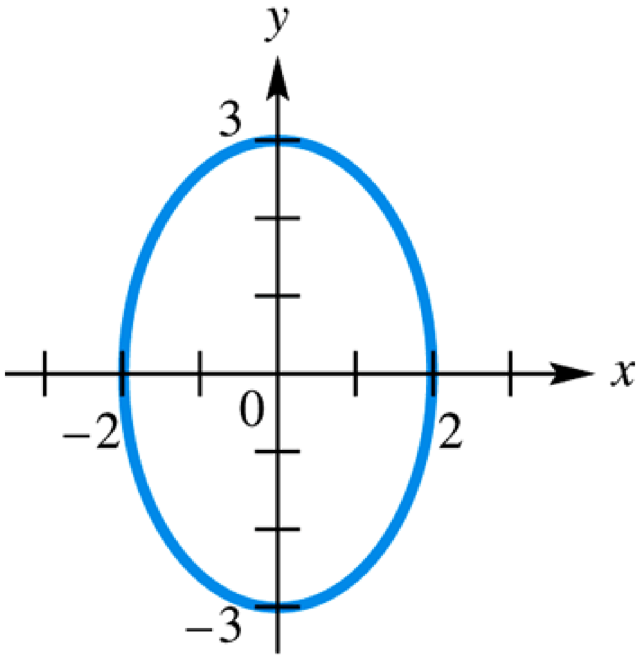
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \quad \text{Substitution}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

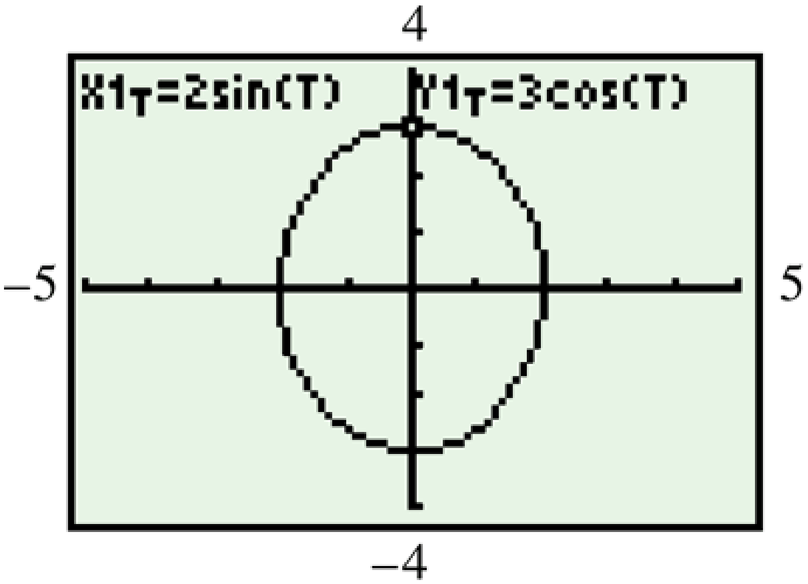
This is an ellipse centered at the origin with axes endpoints (−2, 0), (2, 0), (0, −3), and (0, 3).

Example 3 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY (continued)

$$\left.\begin{aligned} x &= 2\sin t \\ y &= 3\cos t \end{aligned}\right\} \begin{array}{l}\text{for} \\ t \text{ in } [0, 2\pi]\end{array}$$



$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



Parametric graphing mode

▶ Note

Parametric representations of a curve are not unique.

In fact, there are infinitely many parametric representations of a given curve.

Example 4 FINDING ALTERNATIVE PARAMETRIC EQUATION FORMS

Give two parametric representations for the equation of the parabola $y=(x-2)^2+1$.

The simplest choice is

$$x=t,\ y=(t-2)^2+1, \text{ for } t \text{ in } (-\infty,\infty).$$

Another choice is

$$x=t+2,\ y=t^2+1, \text{ for } t \text{ in } (-\infty,\infty).$$

Sometimes trigonometric functions are desirable. One choice is

$$x=\tan t+2,\ y=\sec^2 t, \text{ for } t \text{ in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

The Cycloid

The path traced by a fixed point on the circumference of a circle rolling along a line is called a ***cycloid***.

A cycloid is defined by

$$x = at - a\sin t,\; y = a - a\cos t, \text{ for } t \text{ in } (-\infty, \infty).$$

The Cycloid

If a flexible cord or wire goes through points *P* and *Q*, and a bead is allowed to slide due to the force of gravity without friction along this path from *P* to *Q*, the path that requires the shortest time takes the shape of an inverted cycloid.

Example 5 GRAPHING A CYCLOID

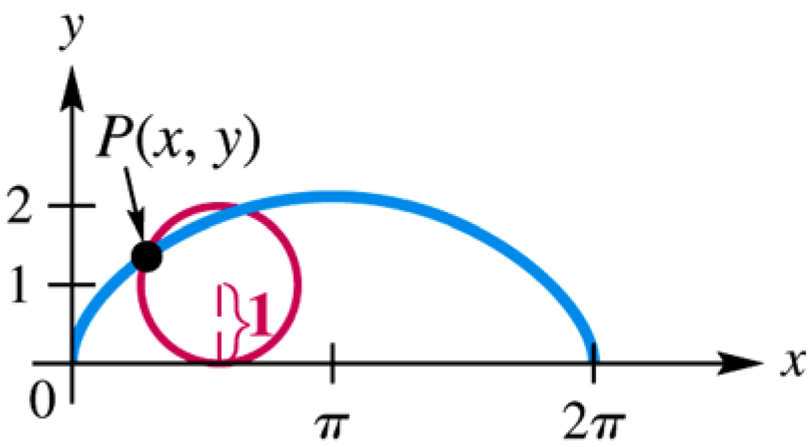
Graph the cycloid $x = t - \sin t$, $y = 1 - \cos t$ for t in $[0, 2\pi]$.

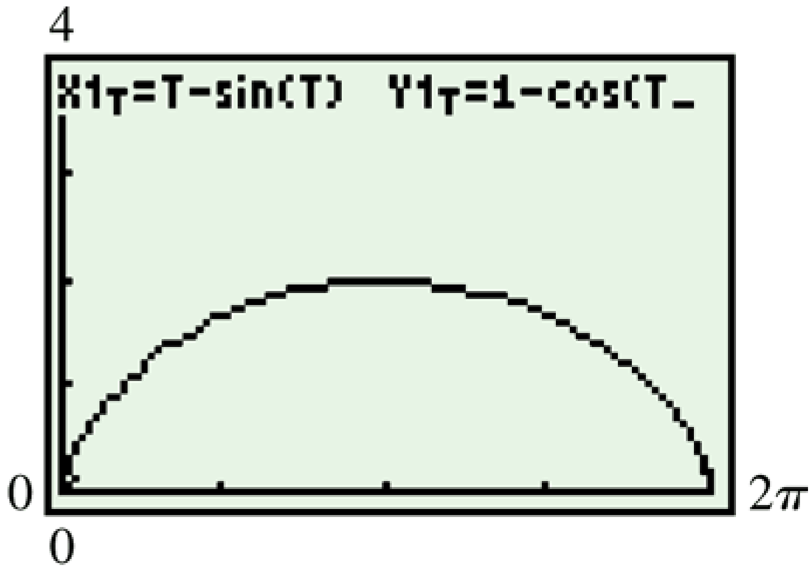
Create a table of values.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$x = t - \sin t$	0	.08	.6	π	5.7	2π
$y = 1 - \cos t$	0	.3	1	2	1	0

Example 5 GRAPHING A CYCLOID (continued)

Plotting the ordered pairs (x, y) from the table of values leads to the portion of the graph for t in $[0, 2\pi]$.



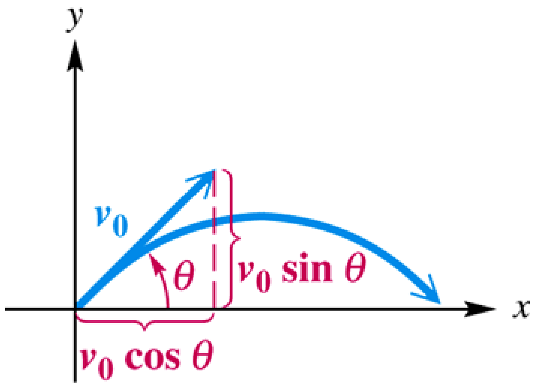


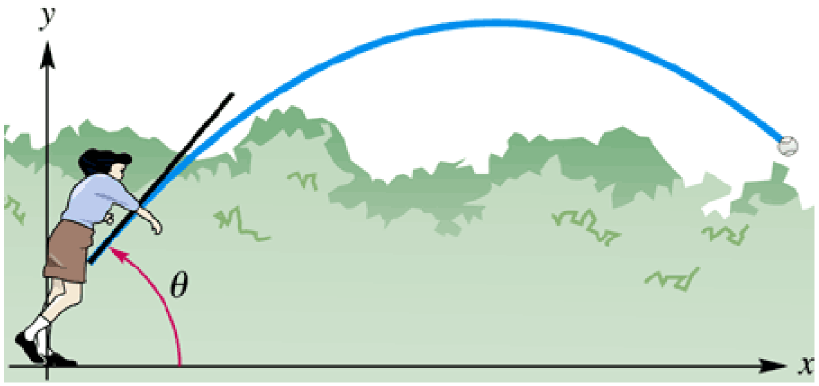
Applications of Parametric Equations

If a ball is thrown with a velocity of *v* feet per second at an angle *θ* with the horizontal, its flight can be modeled by the parametric equations

$$x = (v\cos\theta)t \quad \text{and} \quad y = (v\sin\theta)t - 16t^2 + h,$$

where *t* is in seconds, and *h* is the ball's initial height in feet above the ground.





Example 6 SIMULATING MOTION WITH PARAMETRIC EQUATIONS

Three golf balls are hit simultaneously into the air at 132 feet per second (90 mph) at angles of 30°, 50°, and 70° with the horizontal.

(a) Assuming the ground is level, determine graphically which ball travels the farthest. Estimate this distance.

The three sets of parametric equations determined by the three golf balls ($h = 0$) are

$$x_1 = (132\cos 30°)t,\ y_1 = (132\sin 30°)t - 16t^2$$

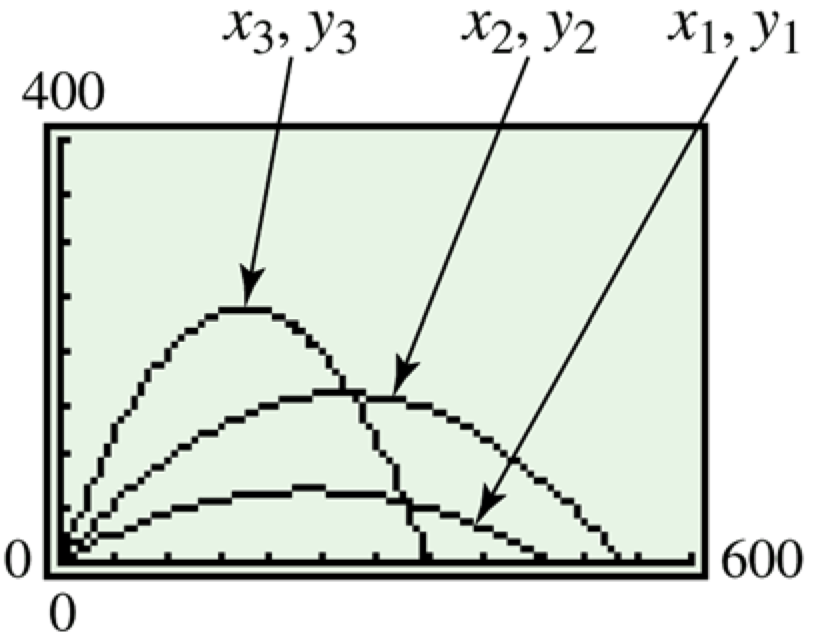
$$x_2 = (132\cos 50°)t,\ y_2 = (132\sin 50°)t - 16t^2$$

$$x_3 = (132\cos 70°)t,\ y_3 = (132\sin 70°)t - 16t^2$$

Example 6

SIMULATING MOTION WITH PARAMETRIC EQUATIONS (continued)

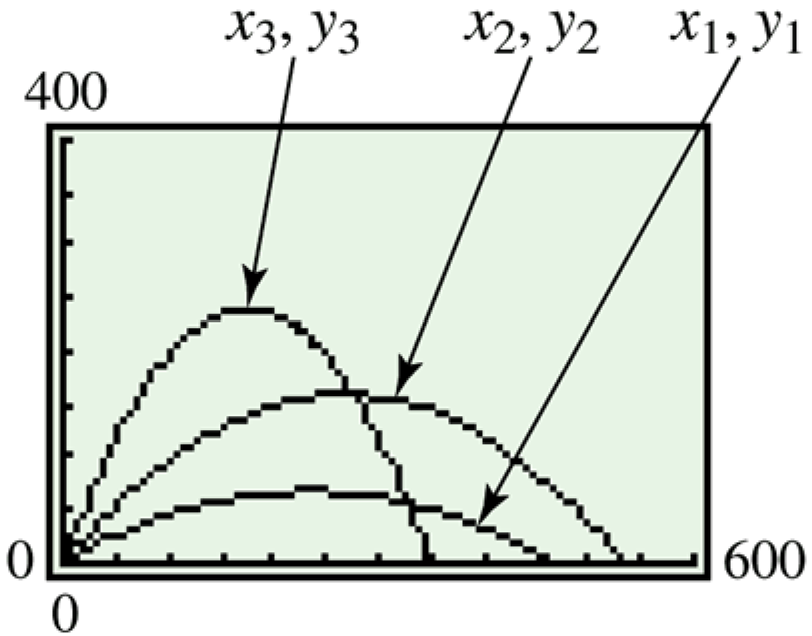
Graph the three sets of parametric equations using a graphing calculator.



Using the TRACE feature, we find that the ball that travels the farthest is the ball hit at 50 . It travels about 540 ft.

▶ Example 6 SIMULATING MOTION WITH PARAMETRIC EQUATIONS (continued)

(b) Which ball reaches the greatest height? Estimate this height.



Using the TRACE feature, we find that the ball that reaches the greatest height is the ball hit at 70 . It reaches about 240 ft.

Example 7 EXAMINING PARAMETRIC EQUATIONS OF FLIGHT

Jack launches a small rocket from a table that is 3.36 ft above the ground. Its initial velocity is 64 ft per sec, and it is launched at an angle of 30 with respect to the ground. Find the rectangular equation that models its path. What type of path does the rocket follow?

The path of the rocket is defined by

$$x = (64\cos 30°)t,\ y = (64\sin 30°)t - 16t^2 + 3.36$$

or equivalently,

$$x = 32\sqrt{3}t,\ y = -16t^2 + 32t + 3.36$$

Example 7 EXAMINING PARAMETRIC EQUATIONS OF FLIGHT (continued)

$$x = 32\sqrt{3}t \Rightarrow t = \frac{x}{32\sqrt{3}}$$

Substituting into the other parametric equation yields

$$y = 16\left(\frac{x}{32\sqrt{3}}\right)^2 + 32\left(\frac{x}{32\sqrt{3}}\right) + 3.36$$

$$y = -\frac{1}{192}x^2 + \frac{\sqrt{3}}{3}x + 3.36$$

Because this equation defines a parabola, the rocket follows a parabolic path.

Example 8 ANALYZING THE PATH OF A PROJECTILE

Determine the total flight time and the horizontal distance traveled by the rocket in Example 7.

From Example 7 (slide 21), we have

$$y = -16t^2 + 32t + 3.36$$

which tells the vertical position of the rocket at time t.

To determine when the rocket hits the ground, solve

$$0 = -16t^2 + 32t + 3.36$$

Example 8 ANALYZING THE PATH OF A PROJECTILE (continued)

$$t = \frac{-32 \pm \sqrt{32^2 - 4(-16)(3.36)}}{2(-16)}$$

$$t = \frac{-32 \pm \sqrt{1239.04}}{-32}$$

$$t = -.1 \text{ (not valid) or } t = 2.1$$

The flight time is about 2.1 seconds.

▶ Example 8 ANALYZING THE PATH OF A PROJECTILE (continued)

Substitute 2.1 for *t* into the parametric equation that models the horizontal position, $x = 32\sqrt{3}t$.

$$x = \left(32\sqrt{3}\right)(2.1) \approx 116.4$$

The rocket has traveled about 116.4 feet.

The solution can be verified by graphing $x = 32\sqrt{3}t$ and $y = -16t^2 + 32t + 3.36$ and using the TRACE feature.